ON THE SUP NORM OF LITTLEWOOD POLYNOMIALS
WITH MAHLER MEASURE ONE ON THE UNIT CIRCLE

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Abstract. Let \( \mathcal{L}_n \) be the collection of all (Littlewood) polynomials of degree \( n \) with coefficients in \( \{-1, 1\} \). In this note we prove that if \( (P_{2\nu}) \) is a sequence of polynomials \( P_{2\nu} \in \mathcal{L}_{2\nu} \) and each zero of each polynomial is on the unit circle, then

\[ M_{2\nu} > (2\nu - 1)^a, \]

where \( a := 1 - \log_3 \frac{\pi}{2} > \frac{1}{2} \) and \( M_{2\nu} \) is the maximum modulus of \( P_{2\nu} \) on the unit circle. A similar result is conjectured for Littlewood polynomials of odd degree. Our main tool here is the Borwein-Choi Factorization Theorem.

1. Introduction

Let \( D \) be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by \( \partial D \). Let \( \mathcal{K}_n \) be the set of all polynomials of degree \( n \) with complex coefficients of modulus 1. Elements of \( \mathcal{K}_n \) are often called (complex) unimodular polynomials of degree \( n \). Let \( \mathcal{L}_n \) be the set of all polynomials of degree \( n \) with coefficients in \( \{-1, 1\} \). Elements of \( \mathcal{L}_n \) are often called real unimodular polynomials or Littlewood polynomials of degree \( n \). The Parseval formula yields

\[ \int_0^{2\pi} |P_n(e^{it})|^2 \, dt = 2\pi(n + 1) \]

for all \( P_n \in \mathcal{K}_n \). Therefore

\[ \min_{z \in \partial D} |P_n(z)| \leq \sqrt{n + 1} \leq \max_{z \in \partial D} |P_n(z)|. \]

An old problem (or rather an old theme) is the following.
Problem 1.1 (Littlewood’s Flatness Problem). How close can a $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ come to satisfying

$$|P_n(z)| = \sqrt{n + 1}, \quad z \in \partial D?$$

Obviously (1.1) is impossible if $n \geq 1$. So one must look for less than (1.1), but then there are various ways of seeking such an “approximate situation”. One way is the following. In his paper [Li1] Littlewood had suggested that, conceivably, there might exist a sequence $(P_n)$ of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that $(n + 1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials “ultraflat”. More precisely, we give the following definition.

**Definition 1.2.** Given a positive number $\varepsilon$, we say that a polynomial $P_n \in \mathcal{K}_n$ is $\varepsilon$-flat if

$$(1 - \varepsilon)\sqrt{n + 1} \leq |P_n(z)| \leq (1 + \varepsilon)\sqrt{n + 1}, \quad z \in \partial D.$$

**Definition 1.3.** Given a sequence $(\varepsilon_{n_k})$ of positive numbers tending to 0, we say that a sequence $(P_{n_k})$ of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is $(\varepsilon_{n_k})$-ultraflat if each $P_{n_k}$ is $(\varepsilon_{n_k})$-flat. We simply say that a sequence $(P_{n_k})$ of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is ultraflat if it is $(\varepsilon_{n_k})$-ultraflat with a suitable sequence $(\varepsilon_{n_k})$ of positive numbers tending to 0.

The existence of an ultraflat sequence of unimodular polynomials seemed very unlikely, in view of a 1957 conjecture of P. Erdős (Problem 22 in [Er]) asserting that, for all $P_n \in \mathcal{K}_n$ with $n \geq 1$,

$$\max_{z \in \partial D} |P_n(z)| \geq (1 + \varepsilon)\sqrt{n + 1},$$

where $\varepsilon > 0$ is an absolute constant (independent of $n$). Yet, refining a method of Körner [Kö], Kahane [Ka] proved that there exists a sequence $(P_{n_k})$ with $P_{n_k} \in \mathcal{K}_{n_k}$ which is $(\varepsilon_{n_k})$-ultraflat, where

$$\varepsilon_n = O\left(n^{-1/17}\sqrt{\log n}\right).$$

See also [QS]. Thus the Erdős conjecture (1.2) was disproved for the classes $\mathcal{K}_n$. For the more restricted class $\mathcal{L}_n$ the analogous Erdős conjecture is unsettled to this date. It is a common belief that the analogous Erdős conjecture for $\mathcal{L}_n$ is true, and consequently there is no ultraflat sequence of polynomials $P_n \in \mathcal{L}_n$. An interesting result related to Kahane’s breakthrough is given in [Be]. For an account of some of the work done till the mid 1960’s, see Littlewood’s book [Li2] and [QS]. The structure of ultraflat sequences of unimodular polynomials is studied in [Er1], [Er2], [Er3], and [Er4], where several conjectures of Saffari are proved.

### 2. New Result

The Rudin-Shapiro polynomials appear in Harold Shapiro’s 1951 thesis at MIT and are sometimes called just Shapiro polynomials. See Chapter 4 of [Bo] for the construction(s). Cyclotomic properties of the Rudin-Shapiro polynomials are discussed in [BLM]. A sequence $(P_n)$ of Rudin-Shapiro polynomials satisfies $P_n \in \mathcal{L}_n$ and

$$|P_n(z)| \leq C\sqrt{n + 1}, \quad z \in \partial D,$$
with an absolute constant $C$. We prove that a sequence of Littlewood polynomials of even
degree with Mahler measure one is far from having the above “flatness” property of a
sequence of Rudin-Shapiro polynomials. Note that (see page 271 of [BE], for instance) a
Littlewood polynomial has Mahler measure one if and only if it has all its zeros on the
unit circle $\partial D$.

**Theorem 2.1.** If $(P_{2\nu})$ is a sequence of polynomials $P_{2\nu} \in \mathcal{L}_{2\nu}$ and each zero of each
polynomial is on the unit circle, then

$$M_{2\nu} > (2\nu - 1)^a,$$

where $a := 1 - \log_{3/2} 2 > \frac{1}{2}$ and $M_{2\nu}$ is the maximum modulus of $P_{2\nu}$ on the unit circle.

It is conjectured that a similar result holds for Littlewood polynomials of odd degree. To prove Theorem 2.1 we need the result from [BC] stated below.

**Theorem 2.2 (Borwein-Choi).** Every polynomial $P \in \mathcal{L}_n$ of even degree can be factor-
ized as

$$P(z) = \pm \Phi_{p_1}(\pm z)\Phi_{p_2}(\pm z^{p_1}) \cdots \Phi_{p_r}(\pm z^{p_1 p_2 \cdots p_r - 1}),$$

where $n - 1 = p_1 p_2 \cdots p_r$, the numbers $p_j$ are primes, not necessarily distinct, and

$$\Phi_p(z) = \sum_{j=0}^{p-1} z^j = \frac{z^p - 1}{z - 1}$$

is the $p$th cyclotomic polynomial.

It is conjectured that this characterization also holds for polynomials $P \in \mathcal{L}_n$ of odd
degree. This conjecture is based on substantial computation together with a number of
special cases.

**Proof of Theorem 2.1.** We use the factorization theorem of Borwein and Choi. We prove
the theorem by induction on the number of factors. To implement the inductive step we
start the numbering of the factors and the corresponding primes from the the end. The
proof of the inductive step goes as follows.

Suppose the theorem is true for $f$, where $f$ has $k - 1$ factors. We have to prove that
theorem is true for

$$g(z) := \Phi_p(\pm z)f(z^p).$$

Let $M(f)$ be the maximum modulus of $f$ on the unit circle. The key observation is that
$M(f)$ is achieved by $|f(z^p)|$ at a system of $p$ equidistant points on the unit circle. Denote
these by $z_1, z_2, \ldots, z_p$. Then there is at least one $z_j$ such that the angular distance between
1 and $z_j$ is at most $2\pi/(2p)$. Similarly there is at least one $z_j$ such that the angular distance
between $-1$ and $z_j$ is at most $2\pi/(2p)$. Now the proof can be finished by Lemma 2.3 below
the proof of which is a straightforward geometric argument. Using Lemma 2.3 the proof
of the inductive step is obvious, since $a := 1 - \log_{3/2} 2 > \frac{1}{2}$ ensures $(2/\pi)p \geq p^a$ for every
$p \geq 3$. In fact, using the prime factorization of $2\nu - 1$, where $2\nu$ is the degree of $P_{2\nu}$, one
can get a larger value of the exponent $a$ in the theorem if the primes in the factorization
of $2\nu + 1$ are large. □
Lemma 2.3. If $z$ is a point on the unit circle such that the angular distance of $z$ from 1 is at most $2\pi/(2p)$, then $|\Phi_p(z)| \geq (2/\pi)p$. If $z$ is a point on the unit circle such that the angular distance of $z$ from $-1$ is at most $2\pi/(2p)$. Then $|\Phi_p(-z)| \geq (2/\pi)p$.

Proof of Lemma 2.3. Recall that

$$\Phi_p(z) = \frac{z^p - 1}{z - 1}$$

and $\sin t \geq (2/\pi)t$ for every $t \in [0, \pi/2]$. □

References


[Li1] J.E. Littlewood, On polynomials $\sum \pm z^m, \sum \exp(\alpha_m i)z^m, z = e^{i\theta}$, J. London Math. Soc. 41, 367–376, yr 1966.


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In the present paper, we study the class LPn which consists of Laurent polynomials $P(z) = (n + 1) + \sum_{k=1}^{n} c_k (z^{k} + z^{-k})$, $k=1, 3, 5, \ldots$.

2. Barker sequences and norms of polynomials on the unit circle

Let $P(z) = a_0 (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n) \in \mathbb{C}[z]$ be a complex polynomial. The Mahler measure of $P(z)$ is defined by

$$M(P) = \max \left\{ |a_0|, \max_{j=1}^{n} |\alpha_j| \right\}.$$ 

In view of Jensen's formula [17], one has

$$\int_{0}^{2\pi} \log |P(e^{it})| \, dt = 2\pi \log \max \left\{ 1, \sup_{|z|=1}|P(z)| \right\}.$$ 

Mahler [16] investigated the maximum of $M(P)$ for polynomials with bounded coefficients. Mahler proved that $M(P)$ is maximized if one takes polynomials $P$ with complex coefficients of equal modulus. Subsequently, Fielding [12], Beller and Newman [2] proved that for such polynomials, the maximum of $M(P)/\|P\|_2$ tends to 1 as the degree $n$ increases to infinity.