ON THE SUP NORM OF LITTLEWOOD POLYNOMIALS WITH MAHLER MEASURE ONE ON THE UNIT CIRCLE

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Abstract. Let $\mathcal{L}_n$ be the collection of all (Littlewood) polynomials of degree n with coefficients in $\{-1, 1\}$. In this note we prove that if $(P_{2\nu})$ is a sequence of polynomials $P_{2\nu} \in \mathcal{L}_{2\nu}$ and each zero of each polynomial is on the unit circle, then

$$M_{2\nu} > (2\nu - 1)^a \,,$$

where $a := 1 - \log_3 \frac{\pi}{2} > \frac{1}{2}$ and $M_{2\nu}$ is the maximum modulus of $P_{2\nu}$ on the unit circle. A similar result is conjectured for Littlewood polynomials of odd degree. Our main tool here is the Borwein-Choi Factorization Theorem.

1. Introduction

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D. Let $\mathcal{K}_n$ be the set of all polynomials of degree n with complex coefficients of modulus 1. Elements of $\mathcal{K}_n$ are often called (complex) unimodular polynomials of degree n. Let $\mathcal{L}_n$ be the set of all polynomials of degree n with coefficients in $\{-1, 1\}$. Elements of $\mathcal{L}_n$ are often called real unimodular polynomials or Littlewood polynomials of degree n. The Parseval formula yields

$$\int_0^{2\pi} |P_n(e^{it})|^2 \, dt = 2\pi(n+1)$$

for all $P_n \in \mathcal{K}_n$. Therefore

$$\min_{z \in \partial D} |P_n(z)| \leq \sqrt{n+1} \leq \max_{z \in \partial D} |P_n(z)| \,.$$

An old problem (or rather an old theme) is the following.

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Problem 1.1 (Littlewood's Flatness Problem). *How close can a $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ come to satisfying*

$$|P_n(z)| = \sqrt{n+1}\,, \qquad z \in \partial D? \tag{1.1}$$

Obviously (1.1) is impossible if $n \geq 1$. So one must look for less than (1.1), but then there are various ways of seeking such an "approximate situation". One way is the following. In his paper [Li1] Littlewood had suggested that, conceivably, there might exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that $(n+1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials "ultraflat". More precisely, we give the following definition.

Definition 1.2. *Given a positive number ε, we say that a polynomial $P_n \in \mathcal{K}_n$ is ε-flat if*

$$(1-\varepsilon)\sqrt{n+1} \leq |P_n(z)| \leq (1+\varepsilon)\sqrt{n+1}\,, \qquad z \in \partial D\,.$$

Definition 1.3. *Given a sequence (ε_{n_k}) of positive numbers tending to 0, we say that a sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is (ε_{n_k})-ultraflat if each P_{n_k} is (ε_{n_k})-flat. We simply say that a sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is ultraflat if it is (ε_{n_k})-ultraflat with a suitable sequence (ε_{n_k}) of positive numbers tending to 0.*

The existence of an ultraflat sequence of unimodular polynomials seemed very unlikely, in view of a 1957 conjecture of P. Erdős (Problem 22 in [Er]) asserting that, for all $P_n \in \mathcal{K}_n$ with $n \geq 1$,

$$\max_{z \in \partial D} |P_n(z)| \geq (1+\varepsilon)\sqrt{n+1}\,, \tag{1.2}$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, refining a method of Körner [Kö], Kahane [Ka] proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n$ which is (ε_n)-ultraflat, where

$$\varepsilon_n = O\left(n^{-1/17}\sqrt{\log n}\right)\,.$$

See also [QS]. Thus the Erdős conjecture (1.2) was disproved for the classes $\mathcal{K}_n$. For the more restricted class $\mathcal{L}_n$ the analogous Erdős conjecture is unsettled to this date. It is a common belief that the analogous Erdős conjecture for $\mathcal{L}_n$ is true, and consequently there is no ultraflat sequence of polynomials $P_n \in \mathcal{L}_n$. An interesting result related to Kahane's breakthrough is given in [Be]. For an account of some of the work done till the mid 1960's, see Littlewood's book [Li2] and [QS]. The structure of ultraflat sequences of unimodular polynomials is studied in [Er1], [Er2], [Er3], and [Er4], where several conjectures of Saffari are proved.

2. New Result

The Rudin-Shapiro polynomials appear in Harold Shapiro's 1951 thesis at MIT and are sometimes called just Shapiro polynomials. See Chapter 4 of [Bo] for the construction(s). Cyclotomic properties of the Rudin-Shapiro polynomials are discussed in [BLM]. A sequence (P_n) of Rudin-Shapiro polynomials satisfies $P_n \in \mathcal{L}_n$ and

$$|P_n(z)| \leq C\sqrt{n+1}\,, \qquad z \in \partial D\,,$$

with an absolute constant C. We prove that a sequence of Littlewood polynomials of even degree with Mahler measure one is far from having the above "flatness" property of a sequence of Rudin-Shapiro polynomials. Note that (see page 271 of [BE], for instance) a Littlewood polynomial has Mahler measure one if and only if it has all its zeros on the unit circle ∂D.

Theorem 2.1. *If $(P_{2\nu})$ is a sequence of polynomials $P_{2\nu} \in \mathcal{L}_{2\nu}$ and each zero of each polynomial is on the unit circle, then*

$$M_{2\nu} > (2\nu - 1)^a \,,$$

where $a := 1 - \log_3 \frac{\pi}{2} > \frac{1}{2}$ and $M_{2\nu}$ is the maximum modulus of $P_{2\nu}$ on the unit circle.

It is conjectured that a similar result holds for Littlewood polynomials of odd degree. To prove Theorem 2.1 we need the result from [BC] stated below.

Theorem 2.2 (Borwein-Choi). *Every polynomial $P \in \mathcal{L}_n$ of even degree can be factorized as*

$$P(z) = \pm\Phi_{p_1}(\pm z)\Phi_{p_2}(\pm z^{p_1}) \cdots \Phi_{p_r}(\pm z^{p_1 p_2 \cdots p_{r-1}})\,,$$

where $n - 1 = p_1 p_2 \cdots p_r$, the numbers p_j are primes, not necessarily distinct, and

$$\Phi_p(z) = \sum_{j=0}^{p-1} z^j = \frac{z^p - 1}{z - 1}$$

is the pth cyclotomic polynomial.

It is conjectured that this characterization also holds for polynomials $P \in \mathcal{L}_n$ of odd degree. This conjecture is based on substantial computation together with a number of special cases.

Proof of Theorem 2.1. We use the factorization theorem of Borwein and Choi. We prove the theorem by induction on the number of factors. To implement the inductive step we start the numbering of the factors and the corresponding primes from the the end. The proof of the inductive step goes as follows.

Suppose the theorem is true for f, where f has $k-1$ factors. We have to prove that theorem is true for

$$g(z) := \Phi_p(\pm z) f(z^p)\,.$$

Let $M(f)$ be the maximum modulus of f on the unit circle. The key observation is that $M(f)$ is achieved by $|f(z^p)|$ at a system of p equidistant points on the unit circle. Denote these by $z_1\,, z_2\,, \ldots\,, z_p$. Then there is at least one z_j such that the angular distance between 1 and z_j is at most $2\pi/(2p)$. Similarly there is at least one z_j such that the angular distance between -1 and z_j is at most $2\pi/(2p)$. Now the proof can be finished by Lemma 2.3 below the proof of which is a straightforward geometric argument. Using Lemma 2.3 the proof of the inductive step is obvious, since $a := 1 - \log_3 \frac{\pi}{2} > \frac{1}{2}$ ensures $(2/\pi)p \geq p^a$ for every $p \geq 3$. In fact, using the prime factorization of $2\nu - 1$, where 2ν is the degree of $P_{2\nu}$, one can get a larger value of the exponent a in the theorem if the primes in the factorization of $2\nu + 1$ are large. $\square$

Lemma 2.3. *If z is a point on the unit circle such that the angular distance of z from 1 is at most $2\pi/(2p)$, then $|\Phi_p(z)| \geq (2/\pi)p$. If z is a point on the unit circle such that the angular distance of z from -1 is at most $2\pi/(2p)$. Then $|\Phi_p(-z)| \geq (2/\pi)p$.*

Proof of Lemma 2.3. Recall that

$$\Phi_p(z) = \frac{z^p - 1}{z - 1}$$

and $\sin t \geq (2/\pi)t$ for every $t \in [0, \pi/2]$. $\square$

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In the present paper, we study the class LPn which consists of Laurent polynomials P (z) = (n + 1) +. n X. ck (z k + z â^'k), k=1 k â€“ odd.Â 2. Barker sequences and norms of polynomials on the unit circle Let P (z) = an (z â^'Î±1)(z â^'Î±2) Â· Â· Â· (z â^'Î±n) â^^ C[z] be a complex polynomial. The Mahler measure of P (z) is defined by n Y M (P) = |a| max {1, |Î±j |} . j=1. In view of Jensenâ€™s formula [17], one has Z 2Ï€ 1 log |P (eit)|dt.Â Mahler [16] investigated the maximum of M (P) for polynomials with bounded coefficients. Mahler proved that M (P) is maximized if one takes polynomials P with complex coefficients of equal modulus. Subsequently, Fielding [12], Beller and Newman [2] proved that for such polynomials, the maximum of M (P)/||P ||2 tends to 1 as the degree n increases to infinity.