
Frame Correspondences in Modal Predicate Logic

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ABSTRACT. Understanding modal predicate logic is a continuing challenge, both philosophical and mathematical. In this paper, I study this system in terms of frame correspondences, finding a number of definability results using substitution methods, including new analyses of axioms in intermediate intuitionistic predicate logics. The semantic arguments often have a different flavour from those in propositional modal logic. But eventually, I hit boundaries to first-order definability of frame conditions. I then relate these findings to the known incompleteness theorems for modal predicate logic, and point out some new directions for further research, including the use of strengthened higher-order proof systems for the basic modal language.¹

1 Introduction

In Grisha Mints' work on modal logic, objects and predication are never far around the corner. That reminds me of my student days, when Hughes and Cresswell 1968 [18] was the reigning textbook, with propositional modal logic only a stepping stone toward modal predicate logic, the vehicle for the real philosophical applications. But gradually, modal propositional logic has stolen the show: in the *Handbook of Modal Logic* (Blackburn, van Benthem & Wolter, eds., 2006 [12]) modal predicate logic gets only one chapter out of twenty-one.² In my own research, van Benthem 1983 [4] does only slightly better, according it one chapter out of nineteen. There are reasons for this change of fortunes. Propositional systems are convenient and application-rich, and their mathematical theory has turned out elegant and challenging. By contrast, since the 1960s, deep difficulties have come to light regarding the very design of modal predicate logic: enough to fill a Black Book.

But my topic here is not dark, but light! I will explore how my original interests in Correspondence Theory cross over from modal propositional logic to modal predicate logic, yielding new theorems and observations.³ The presentation will be fast-paced, geared toward an intended audience of experts reading this book. My feeling is that there is much more to the subject that I broach here, but I must leave that to the reader.

¹This paper is in honor of the many great qualities of my colleague Grisha Mints.

²Garson 2001 [16] and Brauner & Ghilardi 2006 [14] are key references for modal predicate logic. See also the book Gabbay, Shehtman, & Skvortsov 2005 [15], forthcoming soon.

³For Correspondence Theory, van Benthem 1984 [5] is probably still the best source.

2 Modal Predicate Logic: The Basics

We start by recapitulating some well-known notions and results.

DEFINITION 1. The *language of modal predicate logic* arises from the standard formation rules for first-order predicate logic plus a construction clause for the modality, yielding a format

$$Px \mid \neg \mid \vee \mid \wedge \mid \exists x \mid \forall y \mid \diamond \mid \square$$

Formulas with free variables $\varphi = \varphi(x, y, \dots)$ then express modal predication.

One might just combine models here for the two components, producing a family of first-order models ordered by a modal accessibility relation. But a better generalization has turned out to be this:

DEFINITION 2. *Models* for the language of modal predicate logic are structures $\mathbf{M} = (W, R, D, V)$, where W is a set of possible worlds, R an accessibility relation, and D a domain map assigning sets of objects to each possible world. Finally, V is a valuation function interpreting each predicate letter P at each world w as a predicate $V(P, w)$ of the right arity.

Now we must combine the semantics of predicate logic, using assignments taking variables to objects, with the earlier one for modal propositional logic. The following stipulation explains when a formula φ is true at world w under assignment a , where we assume that the values $a(x)$ for the free variables x in φ belong to the domain D_w . Here and in what follows, bold face letters \mathbf{x} (and later on also \mathbf{d}, \mathbf{e}) stand for finite sequences:

$$\begin{aligned} \mathbf{M}, w, a \models P\mathbf{x} & \text{ iff } \text{the tuple of objects } a(\mathbf{x}) \text{ from } D_w \\ & \text{ belongs to the predicate } V(P, w), \\ \mathbf{M}, w, a \models \neg\varphi & \text{ iff } \text{not } \mathbf{M}, w, a \models \varphi \\ \mathbf{M}, w, a \models \varphi \vee \psi & \text{ iff } \mathbf{M}, w, a \models \varphi \text{ or } \mathbf{M}, w, a \models \psi \\ \mathbf{M}, w, a \models \exists x \varphi & \text{ iff } \text{for some } d \in D_w, \mathbf{M}, w, a[x := d] \models \varphi \\ \mathbf{M}, w, a \models \diamond\varphi & \text{ iff } \text{for some } v \text{ with } R w v \text{ where } a(x) \in D_v \text{ for} \\ & \text{all free variables } x \text{ in } \varphi, D, \mathbf{M}, v, a \models \varphi \end{aligned}$$

Here individual quantifiers range over the local domain of the objects that exist at the current world. The clause for the modality makes sure that all objects used by a to evaluate $\diamond\varphi$ in w are also available for evaluating φ in the world v . On the basis of this truth definition, Boolean conjunction \wedge , modal box \square , and universal quantifiers \forall are then defined as usual.

Often, the modal clause of this semantics is simplified by making a further structural assumption that object domains grow along accessibility:

$$\text{For all } w, v, R w v \rightarrow D_w \subseteq D_v \quad \text{Domain Cumulation}$$

These models validate a minimal modal predicate logic fusing standard predicate logic with the minimal propositional modal logic K, where Domain Cumulation ensures validity of the modal distribution axiom. In what follows, we keep this structural property as an optional extra, as it is quite strong. Our

preference is to analyze what axioms mean in terms of frame correspondence, and the weaker the base used then, the better.

3 Translation and Invariance for World-Object Bisimulations

The expressive power of this system can be analyzed with the same techniques that have become standard for modal propositional logic. The first-order correspondence language L_{corr} has two sorts of ‘worlds’ and ‘objects’, basic binary relations Rwv for world accessibility and Ewx for object x being in the domain of world w , and $(k + 1)$ -ary predicates $Pw\mathbf{x}$ for each k -ary predicate $P\mathbf{x}$ in the first-order language of the system.

DEFINITION 3. The *standard translation* $trans(\varphi)$ takes formulas φ in the language of modal predicate logic to L_{corr} -formulas that have the same free object variables as φ plus one free world variable w :

$$\begin{aligned} trans(Px) &= Pwx \\ trans(\neg\varphi) &= \neg trans(\varphi) \\ trans(\varphi \vee \psi) &= trans(\varphi) \vee trans(\psi) \\ trans(\exists x\varphi) &= \exists x(Ewx \wedge trans(\varphi)) \\ trans(\diamond\varphi) &= \exists v(Rwv \wedge \&_i Evx_i(x_i \text{ free in } \varphi) \\ &\quad \wedge [v/w]trans(\varphi)) \end{aligned}$$

Any model \mathbf{M} for modal predicate logic is at the same time a model for the correspondence language L_{corr} , and indeed the following equivalence tightly connects modal semantics with standard first-order evaluation:

THEOREM 4. (Translation Theorem) *For each model \mathbf{M} and each formula φ of modal predicate logic, $\mathbf{M}, w, a \models \varphi$ iff $\mathbf{M}, \alpha \models trans(\varphi)$, where the assignment α sends object variables to their a -values, while the single free world variable of $trans(\varphi)$ goes to the world w .*

Thus, syntactically, modal predicate logic may be seen as a fragment of the full two-sorted first-order language L_{corr} . In this setting, its characteristic semantic invariance can then be defined as a mixture of two well-known structural relations between models: *modal bisimulation*, plus the notion matching it for a full first-order language, namely, *potential isomorphism*.⁴

DEFINITION 5. A *world-object bisimulation* between models \mathbf{M}, \mathbf{N} for modal predicate logic is a relation Z between tuples $w\mathbf{d}$ in \mathbf{M} and $v\mathbf{e}$ in \mathbf{N} of the same length, where all objects in tuples belong to the domain of the initial world.⁵

Here, the relation Z satisfies the following three properties:

⁴Van Benthem 1996 [7] and van Benthem & Bonnay 2008 [10] explore the analogy between potential isomorphism and bisimulation in more general settings.

⁵This assumption will hold throughout this section: when we write a tuple $w\mathbf{d}$, we always assume that the objects mentioned occur in its leading world.

- (a) matching corresponding objects $(\mathbf{d})_i, (\mathbf{e})_i$ of the tuples matched by Z induces a partial isomorphism⁶ between \mathbf{M} and \mathbf{N} ,
- (b) if wRw' in \mathbf{M} and $w'\mathbf{d}$ exists, then there is also a world v' in \mathbf{N} with vRv' and $w'\mathbf{d}Zv'\mathbf{e}$; and the same clause holds in the direction from \mathbf{N} to \mathbf{M} , and
- (c) if d in \mathbf{M} , then there is also an object e in \mathbf{N} with the pair $w\mathbf{d}d, v\mathbf{e}e$ in Z – and again, also vice versa.⁷

In terms of this invariance relation, here is an analogue for modal predicate logic of a well-known characterization for the propositional modal fragment of a pure first-order language over worlds:

THEOREM 6. (Invariance Theorem) *The following statements are equivalent for formulas φ in L_{corr} :*

- (a) φ is invariant for world-object bisimulations,
- (b) φ is definable by a formula of modal predicate logic.

Proof. The crucial step is essentially as that in the Invariance Theorem for propositional modal logic (van Benthem 1996 [7], Theorem 3.12, p. 57; Blackburn, de Rijke & Venema 2000 [13], Theorem 2.68, p. 103). One starts from two models with assignments verifying the same modal predicate-logical formulas (i.e., these models are indistinguishable in the language), and then extends these models to ω -saturated ones. And between models of the later kind, the tuple-to-tuple relation of verifying the same modal predicate-logical formulas turns out to be a world-object bisimulation. ■

4 Frame Correspondences for Special Axioms

Beyond the minimal core, modal axioms impose constraints on models that can be determined by correspondence arguments, as in propositional modal logic. These involve the following notion (van Benthem 1983 [4], Chapter 12):

DEFINITION 7. A formula φ of modal predicate logic *holds in a frame* $\mathbf{F} = (W, R, D)$ (here a ‘frame’ is a model stripped of its valuation) iff φ is true at each world in \mathbf{F} under all valuation functions V .

Examples. Here are some examples, proved by straightforward arguments:

PROPOSITION 8. *The following modal axioms have the listed correspondents:*

$\exists x \diamond Px \rightarrow \diamond \exists x Px$	<i>True</i>	Tautology
$\exists x \Box Px \rightarrow \Box \exists x Px \forall w$	$\forall v (Rwv \rightarrow \forall x (Exw \rightarrow Exv))$	Cumulation
$\diamond \exists x Px \rightarrow \exists x \diamond Px$	$\forall w \forall v (Rwv \rightarrow \forall x (Exv \rightarrow Exw))$	Anti – Cumulation

⁶A *partial isomorphism* between two models is any isomorphism (for the relevant first-order vocabulary) between sub-models of these models – usually *finite* ones.

⁷There are other ways of merging modal bisimulation and potential isomorphism.

Proof. As an illustration, we prove the third correspondence.⁸ Reverse domain inclusion guarantees truth of $\Diamond\exists x Px \rightarrow \exists x \Diamond Px$ in a frame, no matter for which objects in which worlds the predicate P holds. Conversely, let d be any object in any successor world v of w . Make P true only for d in v , and nowhere else. Under this interpretation V , the antecedent $\Diamond\exists x Px$ is true at w , and so by frame truth of the implication for any choice of valuation, under this same V , the world w also has $\exists x \Diamond Px$ true. But then by our truth definition, that can only happen if the object d already exists at w . ■

A general method. Behind this result lies a generalization of the Sahlqvist Theorem for propositional modal logic. Given the right syntactic form of modal axioms, first-order definable equivalents exist on frames, and these can even be computed (van Benthem 1983 [4], Theorem 9.10, p. 105):

THEOREM 9. *There is an effective translation into first-order frame properties for all modal predicate-logical axioms of the syntactic form $\alpha \rightarrow \beta$, where α has the inductive syntax rule $\exists \mid \wedge \mid \vee \mid \Diamond \mid \gamma$, with γ having the simpler syntax $Px \mid \forall \mid \Box$, while β is a wholly positive formula with a syntactic construction rule $Px \mid \forall \mid \exists \mid \wedge \mid \vee \mid \Diamond \mid \Box$.*

Proof. The argument goes just like that for the propositional Sahlqvist theorem via the ‘method of substitutions’ (cf. Blackburn, de Rijke & Venema 2000 [13], Theorem 3.54, p. 165). The crucial point, as explained in these references, is that the special syntactic shape of the modal-quantificational antecedent α allows for first pulling out all existential quantifiers and modalities into a universal prefix, after which the special universal syntactic form of the remaining antecedent formula allows for verification by a first-order definable *minimal* valuation for its atomic predicates.⁹ To get the right first-order frame property for $\alpha \rightarrow \beta$, it suffices to substitute the minimal valuation extracted effectively from α into the standard first-order L_{corr} -translation of the positive consequent formula β . Correctness uses the semantic monotonicity of the latter. ■

Illustrations are earlier quantifier/modality interchanges for $\forall \mid \exists \mid \Box \mid \Diamond$.

Limits. Here is a principle beyond the method of minimal substitutions:

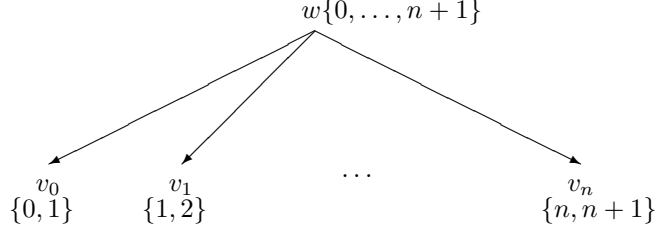
THEOREM 10. $\Box\exists x Px \rightarrow \exists x \Box Px$ has no first-order frame correspondent.

Proof.¹⁰ Consider the following family of finite frames \mathbf{F}_n , each consisting of an irreflexive root world w pointing at n successor worlds v_1, \dots, v_n :

⁸This is a Barcan Axiom that, assuming Cumulation, imposes ‘constant domains’.

⁹Papers [8, 9] state precisely what syntax for first-order antecedents support minimal valuations defining the intersection of all verifying predicates. They also contain extensions to LFP(FO): first-order logic with *fixed-points*.

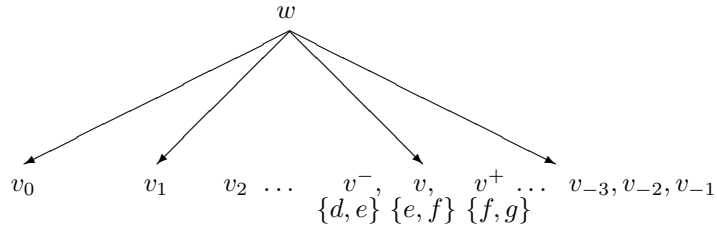
¹⁰What follows is a sketch: the book van Benthem 1983 [4], p. 139, has details.



The domain of w is the finite set of natural numbers $\{0, \dots, n + 1\}$, and each successor world v_i only has two objects in its domain, as follows: v_0 has $\{0, 1\}$, v_1 has $\{1, 2\}$, \dots , v_n has $\{n, n + 1\}$. On each such frame, $\Box \exists x Px \rightarrow \exists x \Box Px$ is true, whatever the valuation for the predicate P . It is clearly true in all the v_i , since these do not have successors. Next, suppose the antecedent $\Box \exists x Px$ holds at w , under an arbitrary valuation. Either object 1 has property P in v_1 , and we get $\Box P1$, since v_1 is the only world where 1 occurs, or it does not. Then 2 has the property P in v_1 , and either it also has P in v_2 , and w has $\Box P2$, or we go on. If we never satisfy $\exists x \Box Px$ in this way, we reach the final point $n + 1$ in v_n as a witness for $\Box P(n + 1)$ at w .

Next, assume that our modal predicate-logical axiom $\Box \exists x Px \rightarrow \exists x \Box Px$ has a first-order frame equivalent α in L_{corr} : we will derive a contradiction. First, we write a set Σ of first-order sentences true in all the above models, describing their main world order and object features. Choose a new relation symbol S imposing an order on the successor worlds. Σ says that S makes the successor worlds v lie in a discrete linear order with a unique beginning and endpoint, and no ‘limit points’. Next, each successor world v contains two objects, while each of these objects occurs in exactly two adjacent worlds – except for the endpoints of S , at each of which one additional isolated object occurs. Also, the objects in the root world are precisely those that occur in some successor world. Finally, we let Σ say that, for each natural number n , there are at least n successor worlds v .¹¹

The above family of frames \mathbf{F}_n clearly shows that the infinite set of formulas $\{\alpha\} \cup \Sigma$ is finitely satisfiable. But then, by Compactness for first-order logic, there is a model \mathbf{M} for all of $\{\alpha\} \cup \Sigma$, with an infinite set of successor worlds v of the following form:



That is, the ordering S of the successor worlds v is like the natural numbers followed by a number of copies of the integers, and ending in a copy of the negative integers. But on such a model, we can refute our modal predicate-logical principle $\Box \exists x Px \rightarrow \exists x \Box Px$ as follows:

¹¹This mouthful is a routine exercise in writing down a lot of first-order formulas.

Let P be false of the isolated objects at the start and the end of the order S , and, using the special domain structure of the worlds v as described, alternating across S -adjacent v -worlds, make P true for just one object in each world in a way that avoids ever giving the same object the property P across two different adjacent worlds.

As a result of this stipulation for predicate P , the antecedent $\Box\exists x Px$ holds in the model \mathbf{M} at the initial world w , but the consequent $\exists x \Box Px$ does not.

But the first-order sentence was true in the model \mathbf{M} by construction, and it was to be equivalent to our modal axiom: a contradiction. ■

Special outcomes in special settings. Here is an interesting counterpoint to known results from the literature. In modal propositional logic, apparently quite mild structural conditions on relational frames can change modal correspondences drastically. In particular, the ‘McKinsey Axiom’

$$\Box\Diamond p \rightarrow \Diamond\Box p$$

is not first-order,¹² but it becomes first-order on transitive frames – where it expresses the property of ‘atomicity’ – though van Benthem 1983 shows that this equivalence cannot be proved by the substitution method. We find a similar effect of seemingly mild structural conditions here:

FACT 11. On frames satisfying Domain Cumulation, the modal predicate-logical axiom $\Box\exists x Px \rightarrow \exists x \Box Px$ is first-order definable.¹³

Proof. The equivalent is the conjunction of two first-order properties:

- (a) Domain Anti-Cumulation,
- (b) each world whose domain has more than one object has at most one world successor.

First, if both of these first-order properties hold in a frame, then so does $\Box\exists x Px \rightarrow \exists x \Box Px$. If a world has just one object d , and we have both Domain Cumulation and Anti-Cumulation, all successors have just that object d , and the antecedent implies that this d has property P throughout. And if a world has at most one accessible successor, then truth of $\Box\exists x Px$ implies that of $\exists x \Box Px$, either trivially since there are no successor worlds, or because some object d in the unique successor world satisfies P , and that same object d will then satisfy $\Box Px$ in w .

Next, we show that frame truth of $\Box\exists x Px \rightarrow \exists x \Box Px$ implies the two stated first-order conditions. First consider Domain Anti-Cumulation (a). Suppose that wRv where v has an object d not occurring in w . We can use any such situation to refute our modal axiom:

In world v , make the predicate P true for d only, and in all other successor worlds of w , make P true for all the objects.

¹²The McKinsey Axiom is not even definable in LFP(FO): cf. [8]. Despite the analogy with $\Box\exists x Px \rightarrow \exists x \Box Px$, the proofs work really differently.

¹³Domain Inclusion crucially did not hold in our preceding counter-example.

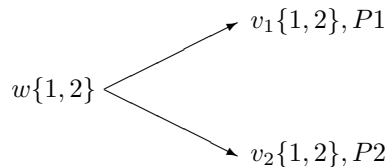
By domain inclusion, the stipulation about world v alone refutes $\exists x \Box Px$ at w , while the two stipulations together make $\Box \exists x Px$ true at w .

Next, take condition (b) of ‘partial function’. Let world w have at least two objects 1, 2 and more than one successor, say v_1 , v_2 and perhaps others. Now define a valuation for the predicate P as follows:

P holds of 1 and of no other object in v_1 , P holds of 2 and no other object in v_2 , and P holds of all objects in all other successor worlds.

This makes the formula $\Box \exists x Px$ true at the world w while there is no object at w that has the property P in all successor worlds. Contradiction, and hence w has at most one successor world. ■

The preceding correspondence argument, though quite elementary, cannot work in the earlier Sahlqvist substitution style. To see this, consider the following model where $\Box \exists x Px \rightarrow \exists x \Box Px$ fails at w :



Given the symmetry between the two successor worlds, there is no uniform definition for the predicate P purely within the language L_{corr} that witnesses this failure. The above correspondence proof in fact used an interpretation for the predicate P that is not definable in this uniform manner.

5 Excursion: Joys of Intuitionistic Predicate Logic

Correspondence analysis is especially vivid in intuitionistic predicate logic. This special area has stronger constraints on frames. Modal accessibility is a *pre-order* satisfying Domain Inclusion, while atomic predicates $P\mathbf{d}$ are *hereditary*: once true at a world, they stay true in all its successor worlds.

Even the propositional version has many surprising features here, witness the thorough study in Rodenburg 1986 [24]. Here is a result on the predicate logic, essentially from Meijer Viol 1995 [20], but in a streamlined presentation.

Intuitionistic semantics gives ‘individuality’ to quantifier principles that are just lumped together as ‘valid’ in classical logic. A nice example is the frame correspondence for what Beth 1959 [11] called ‘Plato’s Law’:¹⁴

$$\exists x (\exists y Py \rightarrow Px) \quad \mathbf{P}$$

Even in the realm of intuitionistic frames, this law defines an interesting second-order mix of purely relational conditions with object occurrence:

THEOREM 12. *On frames, Plato’s Law defines the conjunction of the three conditions: (a) Constant Domains, (b) Accessibility is a linear order if the domain D has at least two objects, (c) If the object domain D is infinite, then the accessibility relation is also well-founded.*

¹⁴Such principles were studied earlier by Casari and Minari: cf. [21].

Proof. First, Plato's Law holds under the stated conditions. For a start, any constant domain with just one object clearly validates the principle. Next, consider a finite domain with objects d_1, \dots, d_n ($n \geq 2$) with a linear world pre-order. Suppose that none of these objects d witnesses Plato's Law: that is, there exists a successor world where property P fails for d while some object other than d satisfies P . But then consider any finite sequence of successor worlds where each object in the initial world finds such a refutation. The minimal world v of this linear order leads to a contradiction. Each object d lacks P in v , by Heredity. But then $\exists y Py$ cannot be true in v to refute Plato's Law for 'its' object d_v . The same argument holds in infinite object domains, by an appeal to the well-foundedness of the linear order.

Conversely, we show that frame truth of Plato's Law implies the stated principles. Consider any world w in a frame. (a) First, if Constant Domain fails, we have a successor world with some new object d not occurring in w . Making P true only for d in w and all its successor worlds in the frame (so as to satisfy Heredity) will refute Plato's Law at w . (b) Next, assume Constant Domain with a set of at least two objects 1, 2, and suppose that Linearity fails: i.e., the world w has two incomparable successors v_1, v_2 . Now make P true of object 1 only in v_1 and all its successor worlds, and likewise for object 2 in v_2 . Again, this satisfies the condition of Heredity while refuting Plato's Law at w . (c) Finally, assume Constant Domain with infinitely many objects in w . Divide up these objects into countably many disjoint sets D_1, D_2, \dots . Now, suppose that the order of the successor worlds violates the condition of well-foundedness. That is, the frame contains a countable descending sequence of successor worlds $\dots v_n R \dots v_2 R v_1$. But then, in the frame, define a predicate P as follows: for each n ,

make P true for the objects in D_n at v_n and all its successor worlds.

This stipulation guarantees Heredity. Moreover, it is easy to verify that with this predicate P , Plato's Law gets refuted at the world w : for each object d in w , the descending sequence has a stage where it still lacks the property P while some other objects already have P . ■

REMARK 1. This correspondence argument has interesting analogies with known ones from modal propositional logic. For instance, the linearity of the accessibility order is also expressed by this consequence:

$$\exists xy \neg x = y \rightarrow ((A \rightarrow B) \vee (B \rightarrow A)),$$

where the consequent can be turned into propositional Sahlqvist form. But *well-foundedness* itself is not definable in propositional modal logic.

6 From Correspondence Arguments to Formal Derivations

Correspondence arguments are semantic and do not necessarily imply the existence of matching formal derivations purely inside some given system of modal (predicate) logic. Indeed, there are some tricky features in setting up the right

proof systems in modal predicate logic. Notoriously, putting together predicate logic with propositional modal logic in an ‘obvious union’ validates a general schema of modal predicate-logical distribution

$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$$

that implies the modal law $\exists x \Box Px \rightarrow \Box \exists x Px$ for Domain Cumulation. This surprising observation from Hughes & Cresswell 1968 [18] shows that innocent-looking combinations of axiom schemata can really be much stronger than merely taking the union of modal logics as ‘theories’.¹⁵

These are not minor issues, and they relate to the discovery of mathematical deficiencies of deduction in modal predicate logic, as early as Ono 1973 [22]. Contributions by many authors, like Shehtman & Skvortsov and Ghilardi, are surveyed in Brauner & Ghilardi 2006 [14]. We will side-step these issues, and just make a few points about formal proofs. First, correspondence arguments like the ones in this paper are proofs of a kind, stated in an informal mathematical meta-language of models – and they can be natural and perspicuous. Thus it makes sense to look for matching deductions in formal systems: in particular, proofs inside modal predicate logic. And indeed, the latter can be really elegant when available.

Here is an illustration, relating Plato’s Law **P** of the preceding section to Grisha Mints’ recent work on intermediate logics relevant to logic programming, presented at the Stanford logic seminar in the spring of 2008.

Excursion: A surprising alternative form of Plato’s Law. Grisha Mints has recently proved that the existential quantifier, not definable in terms of the universal one in intuitionistic logic, does become definable in the modal predicate logic of the two-world Kripke model with constant domains, by the following nice equivalence:

$$\exists x Px \leftrightarrow \forall y ((\forall x (Px \rightarrow Py) \rightarrow Py) \quad \mathbf{M}$$

Here we add a further observation tying this up with known principles:

FACT 13. As a schema over intuitionistic predicate logic, **M** is equivalent to **P**.

Proof. We only need one half of Mints’ principle **M**, viz.

$$\forall y ((\forall x (Px \rightarrow Py) \rightarrow Py) \rightarrow \exists x Px,$$

because the other direction is provable in intuitionistic predicate logic anyway. This is easiest to see when replacing **M** by its intuitionistic equivalent

$$\exists x Px \leftrightarrow \forall y ((\exists x Px \rightarrow Py) \rightarrow Py).$$

¹⁵To see the difference, consider the following ‘proof’ that addition is explicitly definable in first-order arithmetic with 0 and successor S, by a prima facie valid appeal to Beth’s Theorem. Show that addition is implicitly definable, as follows. Consider Peano Arithmetic PA for 0, S plus the usual recursion equations for +. Take a copy PA’ of this theory with 0, S and an operation +’. The combined theory PA + PA’ derives that + and +’ are the same, by an easy induction. E.g., $x + Sy = S(x + y) = S(x +’ y) = x +’ Sy$. But of course, addition is not explicitly definable using just 0 and S! Explanation: we illicitly used the induction axiom of Peano Arithmetic for the *combined language*, which is not in the union of the theories. This shows how powerful schemas can be when used in a richer language.

Staying with the latter version, the relevant half of the equivalence **M** that we need in what follows is

$$\forall y ((\exists x Px \rightarrow Py) \rightarrow Py) \rightarrow \exists x Px.$$

Now here is our equivalence argument:

- (a) From **P** to **M**. Assume that $\forall y ((\exists x Px \rightarrow Py) \rightarrow Py)$ and take a witness for **P**: $\exists y Py \rightarrow Pd$. We get $(\exists x Px \rightarrow Pd) \rightarrow Pd$, and so Pd . But this implies the consequent $\exists x Px$.
- (b) From **M** to **P**. First define $Qu := \exists x Px \rightarrow Pu$. Now plug this into the relevant half of **M**:

$$\begin{aligned} \forall y ((\exists x Qx \rightarrow Qy) \rightarrow Qy) \rightarrow \exists x Qx &= \forall y ((\exists x (\exists x Px \rightarrow Px) \\ &\rightarrow (\exists x Px \rightarrow Py)) \rightarrow (\exists x Px \rightarrow Py)) \rightarrow \exists x (\exists x Px \rightarrow Px). \end{aligned}$$

Finally, it suffices to prove intuitionistically (a routine exercise) that

$$\forall y ((\exists x (\exists x Px \rightarrow Px) \rightarrow (\exists x Px \rightarrow Py)) \rightarrow (\exists x Px \rightarrow Py)).$$

■

But many further results in earlier sections suggest formal proofs.

EXAMPLE 1. Turning correspondence arguments into formal proofs. Assuming Domain Cumulation, the earlier $\Box \exists x Px \rightarrow \exists x \Box Px$ implied Domain Anti-Cumulation, the characteristic property for the Barcan Axiom

$$\forall x \Box Px \rightarrow \Box \forall x Px, \text{ or in an equivalent form: } \Diamond \exists x Px \rightarrow \exists x \Diamond Px.$$

Can we formally derive the latter principle from the former using only the minimal modal predicate logic? This is not clear, as our semantic argument in Section 4 involves first-order substitutions with parameters that are not expressible inside the modal language. Roughly speaking, to closely match that argument, we would need a formal proof proceeding as follows:

“Given that $\Diamond \exists x Px$, pick a successor world v with an object d satisfying the predicate P , and then definably change the denotation of P to P^+ on all other successor worlds to make $\exists x P^+x$ true there. This definition validates $\Box \exists x P^+x$, and hence $\exists x \Box P^+x$, which gives us an object d at w that has the property P^+ in v . But given how we just defined P^+ , this can only mean that d has the old property P at v .”

It is not clear that this proof, no matter how simple, can be made to work with substitutions purely inside the language of modal predicate logic.

Similar challenges arise with deriving the intuitionistic constant domain axiom $\exists xy \neg x = y \rightarrow (\forall x (A \vee \forall x Bx) \rightarrow (A \vee \forall x Bx))$ from Plato’s Law, or with the propositional linearity axiom $\exists xy \neg x = y \rightarrow ((A \rightarrow B) \vee (B \rightarrow A))$.

The general situation is discussed in van Benthem 1979 [3] on incompleteness in modal propositional logic. Correspondence arguments are naturally formalizable in logical systems, but often these proof systems are not purely modal, they need additional second-order comprehension axioms involving first-order definable predicates with object parameters.

FACT 14. Except for the case of Well-foundedness, all correspondence proofs in this paper live in a weak second-order logic with only first-order substitution instances for second-order quantifiers.¹⁶

But van Benthem 1979 [3] also shows that even such simple correspondence arguments relating purely modal axioms via first-order arguments may still lack a counterpart in pure bottom-level modal deduction – by presenting a particularly simple ‘frame-incomplete’ propositional modal logic. In line with this observation, the gap between correspondence and completeness analysis seems even larger in the area of modal predicate logic.

7 Conclusion and Open Problems

We have explored some correspondence theory for modal predicate logic, showing how some promising results can be found. Even so, this foray raises more questions than it answers. Here are a few further directions.

Van Benthem 1983 [4] relates purely relational frame properties definable by axioms in modal predicate logic to such frame properties already defined by purely *propositional modal axioms*, and proves a partial ‘conservativity property’. What is the general link between correspondences in both areas?

Next, the issue of the existence of *formal modal proofs* matching semantic correspondence arguments has merely been raised here, but not analyzed in depth. In the propositional case, this gap has been narrowed using richer ‘hybrid’ modal languages that address the expressive deficit of the modal base language (Venema 1991 [25]). This is also relevant here, and Areces & ten Cate 2006 [1] have the state of the art. Another approach would use yet richer modal languages that arise when analyzing higher-order correspondence arguments. Van Benthem 2006 [9] shows how Löb’s Axiom is frame-definable in first-order logic with *fixed-points* LFP(FO), which can define properties such as the earlier Well-foundedness. It might be interesting to take a similar look at the model predicate-logical case, which can merge fixed-point features of accessibility with recursively defined properties of objects.

Finally, our analysis has worked with just the simplest traditional semantics of modal predicate logic. But as we said at several places, there are open problems in defining this system as a mathematically perspicuous ‘merge’ of modal propositional logic and first-order predicate logic. Indeed, the design of this system has long been under discussion, with many suggested re-modelings making the logic weaker, and hence the gap between validity and modal deduction smaller. Ghilardi proposed category-theoretic ‘functional models’ (van Benthem 1993 [6]; Brauner & Ghilardi 2006 [14]), while Ono 1999 [23] shows

¹⁶‘First-order’ here refers to the *first-order language of frames* with accessibility and elementhood, not to the built-in formal first-order language of objects.

how incompleteness phenomena will sometimes go away when one moves to algebraic models for modal predicate logic. There are also the neighbourhood models of Arlo Costa & Pacuit 2006 [2], or the general frames of Goldblatt & Mares 2006 [17]. In such settings, our correspondence arguments would have to be redone. In this connection, it also seems relevant that predicate logic *itself* may be defined and analyzed as a modal logic (van Benthem 1996 [7], Chapter 9). Thus, modal predicate logic may also be viewed as a product of two modal logics (Kurucz 2006 [19]). Again, our correspondence analysis makes sense here – but it still has to be done.

So few results, so many questions. I will have to join forces with Grisha!

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Knowledge of basic logic—propositional logic, predicate logic—as well as basic mathematics will of course be very helpful. The book treats modal logic as a theory, with several subtheories, such as completeness theory, correspondence theory, duality theory and transfer theory. Thus, the emphasis is on the inner structure of the theory and the connections between the subdisciplines and not on coverage of results. Many problems in modal logic can be solved by constructing polymodal examples and then appealing to this simulation theorem. Chapter 7 discusses the global structure of the lattices of modal logics. The Languages of Description 5.3. Frame Correspondence — An Example 5.4. The Basic Calculus of Internal Descriptions 5.5. Sahlqvist’s Theorem 5.6.