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Leibniz homology of unitary Lie algebras

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0. Introduction

Leibniz homology was introduced by Loday as a noncommutative analog of Lie algebra homology. It can be defined for a broader class of algebras called Leibniz algebras which are nonassociative algebras satisfying the Leibniz identity. Loday and Pirashvili [15] studied the second Leibniz homology group of the Lie algebra $sl_n(S)$ and the Steinberg Lie algebra $\mathfrak{st}_n(S)$, where S is an associative algebra over a commutative ring k. They proved that if S is free as a k-module and $n \geq 5$, then

$$HL_2(sl_n(S)) \cong H_1(S) \quad \text{and} \quad HL_2(\mathfrak{st}_n(S)) \cong \text{Im}B \tag{0.1}$$

where $B : HC_0(S) \to H_1(S)$ is the Connes operator. To do this, they introduced the noncommutative Steinberg algebra $\mathfrak{stl}_n(S)$ and consider the central extension $\rho : \mathfrak{stl}_n(S) \to sl_n(S)$ in the category of Leibniz algebras.

In this paper, we will work on the elementary unitary Lie algebras $eu_n(R, -, \gamma)$ and $eu(\dot{\mathfrak{g}}; R, -)$, where $(R, -)$ is an associative (and commutative in the later case) involutive algebra and $\dot{\mathfrak{g}}$ is a finite dimensional split simple Lie algebra, as was done in [9, 3]. Toward the end, we introduce the noncommutative Steinberg unitary algebra and decompose the Hochschild homology of R by using the involution – on R. We then go on to derive some consequences. For example, we will recover (0.1) for $n \geq 4$ and $\frac{1}{2} \in k$, or $n = 3$ and $\frac{1}{6} \in k$, and $HL_2(\mathfrak{g} \otimes_k S) = \Omega^1_{S|k}$ as well, where $\mathfrak{g}$ is $\dot{\mathfrak{g}}$ or $\dot{\mathfrak{g}}_c$ (the compact form of $\dot{\mathfrak{g}}$) and S is an associative commutative algebra over k.

The paper is organized as follows. In Section 1, we recall some basics on Leibniz algebras, Leibniz homology and various homology theories related to associative algebras. The main reference is the book [14]. Then we study the unitary Lie algebras $eu_n(R, -, \gamma)$ in Section 2 and $eu(\dot{\mathfrak{g}}; R, -)$ in Section 3.

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1. Basics

Let k be a commutative ring. A *Leibniz algebra* L over k is a k-module with a k-bilinear map, called bracket,

$$[\cdot,\cdot] : L \times L \to L$$

satisfying the Leibniz identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y] \tag{1.1}$$

for all $x, y, z \in L$. We sometimes write the Leibniz identity as follows:

$$[[x,y],z] = [x,[y,z]] + [[x,z],y]. \tag{1.2}$$

Clearly, Lie algebras are Leibniz algebras.

Suppose that L is a Leibniz algebra over k. For any $z \in L$, we define $\mathrm{ad}z \in \mathrm{End}_k L$ by

$$(\mathrm{ad}z)x = -[x,z], \quad \text{for all } x \in L. \tag{1.3}$$

It follows from (1.2) that

$$(\mathrm{ad}z)[x,y] = [(\mathrm{ad}z)x, y] + [x, (\mathrm{ad}z)y] \tag{1.4}$$

for all $x, y \in L$. This says that $\mathrm{ad}z$ is a derivation of L. If L is a Lie algebra, then $\mathrm{ad}z$ is just an inner derivation of L defined as usual.

For any Leibniz algebra L there is an associated Lie algebra $L_{\mathrm{Lie}} = L/\langle [x,x] \rangle$ where $\langle [x,x] \rangle$ is the two-sided ideal generated by all $[x,x], x \in L$.

Let L be a Leibniz algebra over k. Consider the tensor product modules,

$$L^{\otimes n} = \underbrace{L \otimes_k \cdots \otimes_k L}_{n},$$

one has the boundary map: $d_n : L^{\otimes n} \to L^{\otimes (n-1)}$ defined by

$$d_n(g_1 \otimes \cdots \otimes g_n) = \sum_{1 \le i < j \le n} (-1)^{j+1} g_1 \otimes \cdots \otimes g_{i-1} \otimes [g_i, g_j] \otimes g_{i+1} \otimes \cdots \otimes \hat{g}_j \otimes \cdots \otimes g_n,$$

where $\hat{g}_j$ indicates that the term g_j is omitted. One can show that $d^2 = 0$ and the complex $(L^{\otimes n}, d)$ ($L^0 = k$ and $d_1 = 0$) gives the Leibniz homology $HL_*(L)$ of the Leibniz algebra L. Note that only the Leibniz identity is needed to guarantee $d^2 = 0$. This is actually the motivation for definitions of Leibniz algebras and their homology (see [14, 15]).

Let L be a Leibniz algebra over k. The *center* of L is defined to be $\{z \in L : [z, L] = [L, z] = (0)\}$. L is called *perfect* if $[L, L] = L$. A *central extension* of L is a pair $(\hat{L}, \pi)$ where $\hat{L}$ is a Leibniz algebra and $\pi : \hat{L} \to L$ is a surjective homomorphism such that

$\ker \pi$ lies in the center of $\hat{L}$ and the exact sequence $0 \to \ker \pi \to \hat{L} \to L \to 0$ splits as k-modules. The pair $(\hat{L}, \pi)$ is a *universal central extension* of L if for every central extension $(\tilde{L}, \tau)$ of L there is a unique homomorphism $\psi : \hat{L} \to \tilde{L}$ for which $\tau \circ \psi = \pi$. So the universal central extension is unique, up to isomorphism.

The following result is also known.(see [15]).

Proposition 1.1. *The universal central extension of a Leibniz algebra L exists if and only if L is perfect. If $(\hat{L}, \pi)$ is the universal central extension of L, then $HL_2(L) \cong \ker \pi$.*

Let R be an associative k-algebra with identity 1. Consider the tensor product modules over k,

$$R^{\otimes(n+1)} = \underbrace{R \otimes_k R \otimes \cdots \otimes_k R}_{n+1},$$

one has the Hochschild boundary: $d_n : R^{\otimes(n+1)} \to R^{\otimes n}$ defined by

$$\begin{aligned} &d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \\ &\quad = \left(\sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \right) + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}, \end{aligned}$$

and knows that $d^2 = 0$ where $R^{\otimes 0} = k$ and $d_0 = 0$. The complex $(R^{\otimes(*+1)}, d)$ gives the Hochschild homology $H_*(R)$ of the associative algebra R.

Throughout the rest of this paper, we assume that $\frac{1}{2} \in k$.

Suppose that R is, in addition, equipped with an (anti)-involution $^-$, we write $(R, -)$ and denote $R_+ = \{a \in R : \bar{a} = a\}$ and $R_- = \{a \in R : \bar{a} = -a\}$.

Let $\mathbb{D}_{n+1} = \langle t_{n+1}, l_{n+1} \mid l_{n+1}^2 = t_{n+1}^{n+1} = 1,\ l_{n+1} t_{n+1} l_{n+1} = t_{n+1}^{-1} \rangle$ be the dihedral group and define an action on the k-module $R^{\otimes(n+1)}$ as follows:

$$l_{n+1}(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^{n(n+1)/2} \bar{a}_0 \otimes \bar{a}_n \otimes \bar{a}_{n-1} \otimes \cdots \otimes \bar{a}_1,$$

$$t_{n+1}(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots a_{n-1}$$

for $a_i \in R, 0 \le i \le n$. Let

$$C_n(R) = \frac{R^{\otimes(n+1)}}{\mathrm{Im}(t_{n+1} - 1)},$$

$$D_n(R) = \frac{R^{\otimes(n+1)}}{\mathrm{Im}(l_{n+1} - 1) + \mathrm{Im}(t_{n+1} - 1)},$$

$$_{-1}D_n(R) = \frac{R^{\otimes(n+1)}}{\mathrm{Im}(l_{n+1} + 1) + \mathrm{Im}(t_{n+1} - 1)},$$

$$L_n(R) = \frac{R^{\otimes(n+1)}}{\mathrm{Im}(l_{n+1} - 1)} \quad \text{and} \quad {}_{-1}L_n(R) = \frac{R^{\otimes(n+1)}}{\mathrm{Im}(l_{n+1} + 1)}$$

be quotient modules. The Hochschild boundary d induces boundary maps for each of the five complexes $(C_*(R),d),(D_*(R),d),({}_{-1}D_*(R),d),(L_*(R),d)$ and $({}_{-1}L_*(R),d)$, the corresponding homology is denoted by $HC_*(R),HD_*(R),{}_{-1}HD_*(R),H_*^+(R)$ and $H_*^-(R)$, respectively. The first three are called the cyclic, dihedral and skew-dihedral homology, respectively. For more about this, see [13, 14].

It is easy to see that

$$HC_0(R)=\frac{R}{[R,R]},\quad HD_0(R)=\frac{R}{[R,R]+R_-}\quad \text{and}\quad {}_{-1}HD_0(R)=\frac{R}{[R,R]+R_+}.$$

For our use we rewrite $H_1(R),H_1^-(R),HC_1(R)$ and ${}_{-1}HD_1(R)$ as was done in [9].

Proposition 1.2. *Let $\langle R,R\rangle=(R\otimes_k R)/I$ be the quotient module, where I is the submodule of $R\otimes_k R$ generated by elements $ab\otimes c-a\otimes bc+ca\otimes b=d_2(a\otimes b\otimes c)$ for all $a,b,c\in R$. Set $\langle a,b\rangle=a\otimes b+I$, then*

$$H_1(R)=\left\{\sum_i\langle a_i,b_i\rangle \mid \sum_i(a_ib_i-b_ia_i)=0\right\}$$

is a submodule of $\langle R,R\rangle$.

Proposition 1.3. *Let $\langle R,R\rangle_c=(R\otimes_k R)/I_c$ be the quotient module, where I_c is the submodule of $R\otimes_k R$ generated by elements $ab\otimes c-a\otimes bc+ca\otimes b, a\otimes b+b\otimes a$ for all $a,b,c\in R$. Set $\langle a,b\rangle_c=a\otimes b+I_c$, then*

$$HC_1(R)=\left\{\sum_i\langle a_i,b_i\rangle_c \mid \sum_i(a_ib_i-b_ia_i)=0\right\}$$

is a submodule of $\langle R,R\rangle_c$.

Now, consider $(R,-)$. It is easy to see that $a=0 \pmod{R_+}$ if and only if $a=\bar{a}$.

Proposition 1.4. *Let $\langle R,R\rangle^-=(R\otimes_k R)/I^-$ be the quotient module, where I^- is the submodule of $R\otimes_k R$ generated by elements $a\otimes b-\bar{a}\otimes\bar{b}, ab\otimes c-a\otimes bc+ca\otimes b$ for all $a,b,c\in R$. Set $\langle a,b\rangle^-=a\otimes b+I^-$, then*

$$H_1^-(R)=\left\{\sum_i\langle a_i,b_i\rangle^- \mid \sum_i\overline{a_ib_i-b_ia_i}=\sum_i(a_ib_i-b_ia_i)\right\}$$

is a submodule of $\langle R,R\rangle^-$.

Proposition 1.5. *Let $\langle R,R\rangle_d=(R\otimes_k R)/I_d$ be the quotient module, where I_d is the submodule of $R\otimes_k R$ generated by elements $a\otimes b-\bar{a}\otimes\bar{b}, ab\otimes c-a\otimes bc+ca\otimes b, a\otimes b+b\otimes a$ for all $a,b,c\in R$. Set $\langle a,b\rangle_d=a\otimes b+I_d$, then*

$${}_{-1}HD_1(R)=\left\{\sum_i\langle a_i,b_i\rangle_d \mid \sum_i\overline{a_ib_i-b_ia_i}=\sum_i(a_ib_i-b_ia_i)\right\}$$

is a submodule of $\langle R,R\rangle_d$.

We denote the quotient maps from $R \otimes_k R$ to $\langle R,R\rangle, \langle R,R\rangle_c, \langle R,R\rangle^-$ and $\langle R,R\rangle_d$ by q, q_c, q^- and q_d, respectively. We also denote the natural map $\langle R,R\rangle \to \langle R,R\rangle_c$ (or $\langle R,R\rangle^- \to \langle R,R\rangle_d$) by p (or p^-). Clearly,

$$p \circ q = q_c \qquad \text{and} \qquad p^- \circ q^- = q_d. \tag{1.6}$$

Remark 1.6. If R is commutative, then $H_1(R) = \Omega^1_{R|k}$, the Kähler differentials (see [14] for the definition). Moreover, when the involution $^-$ is trivial, one has

$$H_1^-(R) = H_1(R) = \Omega^1_{R|k}.$$

The following result is an analogue of Proposition 1.13 of [9]

Proposition 1.7. *Assume that S is an associative k-algebra with identity. Let S^{op} be its opposite algebra. Let $R = (S \oplus S^{\mathrm{op}}, ex)$, where $ex = -$ and $\overline{(s,s')} = (s',s)$, then $\varphi^- : \langle R,R\rangle^- \to \langle S,S\rangle$ given by*

$$\varphi^-(\langle (a_1,a_2),(b_1,b_2)\rangle^-) = \langle a_1,b_1\rangle + \langle a_2,b_2\rangle$$

is an isomorphism and $H_1^-(R) \cong H_1(S)$.

Suppose that -1 has no square root in k. Let S be an associative commutative k-algebra with identity, and $R = S \otimes_k k(i) = S \oplus \mathrm{i}S$, where $\mathrm{i} = \sqrt{-1}$. Define $^- : R \to R$ by $\overline{a + \mathrm{i}b} = a - ib$ for all $a,b \in S$, then R is an associative commutative k-algebra equipped with an involution $^-$. In this case we claim

Proposition 1.8. $H_1^-(R) \cong H_1(S)$.

Proof. Define $f : R \otimes R \to H_1(S)$ by

$$f((a_1 + \mathrm{i}b_1) \otimes (a_2 + \mathrm{i}b_2)) = \langle a_1,a_2\rangle - \langle b_1,b_2\rangle.$$

Then one can show that the following elements:

$$\begin{aligned}&(a_1 + \mathrm{i}b_1) \otimes (a_2 + \mathrm{i}b_2) - (a_1 - \mathrm{i}b_1) \otimes (a_2 - \mathrm{i}b_2),\\ &\quad (a_1 + \mathrm{i}b_1)(a_2 + \mathrm{i}b_2) \otimes (a_3 + \mathrm{i}b_3) - (a_1 + \mathrm{i}b_1) \otimes (a_2 + \mathrm{i}b_2)(a_3 + \mathrm{i}b_3)\\ &\quad +(a_3 + \mathrm{i}b_3)(a_1 + \mathrm{i}b_1) \otimes (a_2 + \mathrm{i}b_2),\end{aligned}$$

lie in the kernel of f, therefore f induces a surjective homomorphism

$$f : H_1^-(R) \to H_1(S).$$

In $H_1^-(R)$, we have $\langle \mathrm{i}b,a\rangle^- = 0$ and $\langle a,\mathrm{i}b\rangle^- = 0$. Next,

$$\langle \mathrm{i}\cdot a, \mathrm{i}b\rangle^- - \langle \mathrm{i}, a\cdot \mathrm{i}b\rangle^- + \langle \mathrm{i}b\cdot \mathrm{i}, a\rangle^- = 0,$$

which yields

$$\langle \mathrm{i}a, \mathrm{i}b\rangle^- - \langle \mathrm{i}, ab\mathrm{i}\rangle^- - \langle b,a\rangle^- = 0. \tag{1.7}$$

Also, $\langle a, 1\rangle^- = 0$, and $\langle ab \cdot \mathrm{i}, \mathrm{i}\rangle^- - \langle ab, \mathrm{i}\cdot \mathrm{i}\rangle^- + \langle \mathrm{i}\cdot ab, \mathrm{i}\rangle^- = 0$ give us $\langle ab\mathrm{i}, \mathrm{i}\rangle^- = 0$. It then follows from

$$\langle \mathrm{i}\cdot ab, \mathrm{i}\rangle^- - \langle \mathrm{i}, ab\cdot \mathrm{i}\rangle^- + \langle \mathrm{i}\cdot \mathrm{i}, ab\rangle^- = 0,$$

that $\langle \mathrm{i}, ab\mathrm{i}\rangle^- = -\langle 1, ab\rangle^-$. From (1.7) we have

$$\langle \mathrm{i}a, \mathrm{i}b\rangle^- = \langle \mathrm{i}, ab\mathrm{i}\rangle^- + \langle b, a\rangle^- = -\langle 1, ab\rangle^- + \langle b, a\rangle^- = -\langle a, b\rangle^-$$

and

$$\langle a_1 + \mathrm{i}b_1, a_2 + \mathrm{i}b_2\rangle^- = \langle a_1, a_2\rangle^- - \langle b_1, b_2\rangle^- .$$

Now, we define $g : S\otimes S \to H_1^-(R)$ by $g(a\otimes b) = \langle a, b\rangle^-$. Obviously, g induces an epimorphism $g : H_1(S) \to H_1^-(R)$ such that $g(\langle a, b\rangle) = \langle a, b\rangle^-$. Then

$$\begin{aligned} g\circ f(\langle a_1 + \mathrm{i}b_1, a_2 + \mathrm{i}b_2\rangle^-) &= g(\langle a_1, a_2\rangle - \langle b_1, b_2\rangle) \\ &= \langle a_1, a_2\rangle^- - \langle b_1, b_2\rangle^- = \langle a_1 + \mathrm{i}b_1, a_2 + \mathrm{i}b_2\rangle^-, \end{aligned}$$

which shows that $g\circ f = id$ and that f is also injective. □

Next, we want to investigate the kernel of p^-.

Lemma 1.9. (i) $\langle 1, [R, R] + R_-\rangle^- = 0$. (ii) $\ker\ p^- = \langle 1, R\rangle^-$.

Proof. Since

$$\begin{aligned} &\langle b, c\rangle^- - \langle 1, bc\rangle^- + \langle c, b\rangle^- = 0, \\ &\langle c, b\rangle^- - \langle 1, cb\rangle^- + \langle b, c\rangle^- = 0, \end{aligned} \tag{1.8}$$

we have $\langle 1, bc - cb\rangle^- = 0$, for all $b, c \in R$. Also, $\langle 1, b\rangle^- - \langle 1, \bar{b}\rangle^- = 0$, for all $b\in R$. So (i) holds.

Assume that $p^-(u) = 0$ for some $u \in \langle R, R\rangle^-$. Since q^- is onto, we have $u = q^-(v)$ for some $v \in R\otimes_k R$. Then $q_d(v) = p^-\circ q^-(v) = p^-(u) = 0$ which says that $v \in \ker\ q_d$. Thus,

$$v = \sum_i (a_ib_i \otimes c_i - a_i \otimes b_ic_i + c_ia_i \otimes b_i) + \sum_j (d_j \otimes e_j + e_j \otimes d_j)$$

for some $a_i, b_i, c_i, d_j, e_j \in R$. So $q^-(v) = \sum_j \langle d_j, e_j\rangle^- + \langle e_j, d_j\rangle^-$. It then follows from (1.8) that

$$u = q^-(v) = \sum_j \langle 1, d_je_j\rangle^- = \langle 1, \sum_j d_je_j\rangle^-$$

as needed. □

Now, we can define

$$\mathfrak{B} : HD_0(R) = \frac{R}{[R,R] + R\_} \to \langle R,R\rangle^-, \tag{1.9}$$

by $\mathfrak{B}(a + [R,R] + R\_) = \langle 1,a\rangle^-$.

Proposition 1.10. $\operatorname{Im}\mathfrak{B} = \ker p^- \subseteq H_1^-(R)$ *and we have the following exact sequence*

$$HD_0(R) \xrightarrow{\mathfrak{B}} H_1^-(R) \xrightarrow{\varphi^-} {}_{-1}HD_1(R).$$

Proof. Clearly, Im $\mathfrak{B} = \ker p^- \subseteq H_1^-(R)$ and the following sequence is exact:

$$HD_0(R) \xrightarrow{\mathfrak{B}} \langle R,R\rangle^- \xrightarrow{\varphi^-} \langle R,R\rangle_d.$$

Taking restriction gives the desired exact sequence. □

Remark 1.11. If R is commutative and the involution - is trivial, then $H_1^-(R) = \Omega^1_{R|k}$ and Im $\mathfrak{B} = dR$ are just exact forms in $\Omega^1_{R|k}$.

Remark 1.12. If $(R,-) = (S \oplus S^{\mathrm{op}}, ex)$ as in Proposition 1.7, then the following two exact sequences are isomorphic:

$$\begin{array}{ccccc} HD_0(R) & \xrightarrow{\mathscr{B}} & H_1^-(R) & \xrightarrow{p^-} & {}_{-1}HD_1(R) \\ \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi^-} & & \downarrow{\scriptstyle \varphi_1} \\ HC_0(S) & \xrightarrow{B} & H_1(S) & \xrightarrow{p} & HC_1(S), \end{array}$$

where $\varphi_1(\langle(a_1,a_2),(b_1,b_2)\rangle_d) = \langle a_1,b_1\rangle_c + \langle a_2,b_2\rangle_c$ (see [9]), $\varphi_0((a,b)+[R,R]+R\_) = a+b+[S,S]$ and B is the so-called Connes operator (see [16] or [14]).

To conclude this section, we need a bit more notations.

Let J be the submodule of $R \otimes_k R$ generated by the following elements:

$$a \otimes b + b \otimes a,\ \bar{a}b \otimes c - a \otimes \bar{b}c + \bar{c}a \otimes b,$$

$$(c(ab - \overline{ab}) + (ba - \overline{ba})c) \otimes d$$

for all $a,b,c,d \in R$. Define $\mathscr{L}(R,-) = (R \otimes_k R)/J$ to be the quotient k-module and write $\ell(a,b) = a \otimes b + J$ (see [3]). Let $r : R \otimes_k R \to \mathscr{L}(R,-)$ be the quotient map.

Now, we introduce a noncommutative analog of $\mathscr{L}(R,-)$. Let N be the submodule of $R \otimes_k R$ spanned by the following elements:

$$\bar{a}b \otimes c - a \otimes \bar{b}c + \bar{c}a \otimes b,$$

$$(c(ab - \overline{ab}) + (ba - \overline{ba})c) \otimes d,\ d \otimes (c(ab - \overline{ab}) + (ba - \overline{ba})c)$$

for all $a,b,c,d \in R$. Define $\mathscr{N}(R,-) = (R \otimes_k R)/N$ to be the quotient k-module and write $\ell\_(a,b) = a \otimes b + N$. Let $\hat{r} : R \otimes_k R \to \mathscr{N}(R,-)$ be the quotient map. There is a natural map $Q:\mathcal{N}(R,-)\to\mathcal{L}(R,-)$ given by $Q(\ell\_(a,b))=\ell(a,b)$ such that $Q\circ\hat{r}=r$. As Lemma 1.9, one can show that

Lemma 1.13. (i) $\ell\_(1,[R,R]+R\_)=0$. (ii) $\ker Q=\ell\_(1,R)$.

Define $P:HD_0(R)\to\mathcal{N}(R,-)$ by $P(a+[R,R]+R\_)=\ell\_(1,a)$. Then

$$\ker Q=\ \mathrm{Im}P. \tag{1.10}$$

Remark 1.14. One can further prove that $\ell\_(1,RR\_+R\_R)=0$.

2. Noncommutative Steinberg unitary algebras

Let R be an associative k-algebra with identity. The k-Lie algebra of $n\times n$ matrices with coefficients in R is denoted by $gl_n(R)$. For $n\geq 2$, the elementary Lie algebra $sl_n(R)$ (or $e_n(R)$) is the subalgebra of $gl_n(R)$ generated by the elements $e_{ij}(a)$, $a\in R$, $1\leq i\neq j\leq n$, where e_{ij} are standard matrix units.

For $n\geq 3$, the *noncommutative Steinberg algebra* $\mathfrak{stl}_n(R)$ is defined to be the Leibniz algebra over k generated by the symbols $X_{ij}(a)$, $a\in R$, $1\leq i\neq j\leq n$, subject to the relations (see [15]):

$$a\mapsto X_{ij}(a)\ \text{is a } k\text{-linear mapping}, \tag{2.1}$$

$$[X_{ij}(a),X_{jk}(b)]=-[X_{jk}(b),X_{ij}(a)]=X_{ik}(ab),\ \text{for distinct } i,j,k, \tag{2.2}$$

$$[X_{ij}(a),X_{kl}(b)]=0,\ \text{for } j\neq k, i\neq l, \tag{2.3}$$

where $a,b\in R$, $1\leq i,j,k,l\leq n$. Let $\rho:\mathfrak{stl}_n(R,-,\gamma)\to sl_n(R)$ be the map defined by $\rho(X_{ij}(a))=e_{ij}(a)$. Then ρ yields a central extension.

If R is, in addition, equipped with an (anti)-involution $^-$, then the elementary unitary Lie algebra $eu_n(R,-,\gamma)$ is a subalgebra of $gl_n(R)$ generated by the elements $e_{ij}(a)-\gamma_i\gamma_j^{-1}e_{ji}(\bar{a})$, $a\in R$, $1\leq i\neq j\leq n$, where $\gamma=(\gamma_1,\ldots,\gamma_n)$, $\gamma_i\in k^\times$, the units of k, $1\leq i\leq n$.

It is easy to see that $eu_n(R,-,\gamma)$ has a k-module decomposition

$$eu_n(R,-,\gamma)=\xi_0\oplus\sum_{1\leq i<j\leq n}\oplus\xi_{ij}, \tag{2.4}$$

where $\xi_{ij}=\{e_{ij}(a)-\gamma_i\gamma_j^{-1}e_{ji}(\bar{a})|a\in R\}$ for $1\leq i<j\leq n$ and ξ_0 is the subalgebra of diagonal matrices of $eu_n(R,-,\gamma)$, which is spanned by the elements

$$[e_{ij}(a)-\gamma_i\gamma_j^{-1}e_{ji}(\bar{a}),e_{ji}(b)-\gamma_j\gamma_i^{-1}e_{ij}(\bar{b})]=e_{ii}(ab-\overline{ab})+e_{jj}(\overline{ba}-ba) \tag{2.5}$$

for $a,b\in R$, $1\leq i\neq j\leq n$.

For later use, we consider $eu_3(R,-,\gamma)$ when R is commutative and the involution $^-$ is the identity map.

Lemma 2.1. *If* $\gamma = (1,1,1)$, *then*

$$eu_3(R,-,\gamma) \cong k^3 \otimes R,$$

where k^3 *is the three-dimensional Lie algebra over* k *with a basis* $\{x,y,z\}$ *and relations* $[x,y] = z, [y,z] = x, [z,x] = y$.

Proof. The map

$$\begin{aligned} e_{12}(a) - e_{21}(a) &\mapsto x \otimes a, \\ e_{23}(a) - e_{32}(a) &\mapsto y \otimes a, \\ e_{13}(a) - e_{31}(a) &\mapsto z \otimes a, \end{aligned}$$

gives the isomorphism of two algebras. □

Lemma 2.2. *If* $\gamma = (1,-1,1)$, *then*

$$eu_3(R,-,\gamma) \cong sl_2(k) \otimes R$$

where $sl_2(k)$ *is the three-dimensional Lie algebra over* k *with a basis* $\{e,f,h\}$ *and relations* $[h,e] = 2e, [h,f] = -2f, [e,f] = h$.

Proof. The map

$$\begin{aligned} e_{12}(a) + e_{21}(a) &\mapsto \tfrac{1}{2} h \otimes a, \\ e_{23}(a) + e_{32}(a) + e_{13}(a) - e_{31}(a) &\mapsto e \otimes a, \\ e_{23}(a) + e_{32}(a) - (e_{13}(a) - e_{31}(a)) &\mapsto f \otimes a, \end{aligned}$$

gives the desired isomorphism. □

For $n \geq 3$, the *noncommutative Steinberg unitary algebra* $\mathfrak{stul}_n(R,-,\gamma)$ is defined to be the Leibniz algebra over k generated by the symbols $u_{ij}(a)$, $a \in R$, $1 \leq i \neq j \leq n$, subject to the relations:

$$u_{ij}(a) = u_{ji}(-\gamma_i \gamma_j^{-1} \bar{a}), \tag{2.6}$$

$$a \mapsto u_{ij}(a) \text{ is a } k\text{-linear mapping}, \tag{2.7}$$

$$[u_{ij}(a), u_{jk}(b)] = -[u_{jk}(b), u_{ij}(a)] = u_{ik}(ab), \text{ for distinct } i,j,k, \tag{2.8}$$

$$[u_{ij}(a), u_{kl}(b)] = 0, \quad \text{for distinct } i,j,k,l, \tag{2.9}$$

where $a,b \in R$, $1 \leq i,j,k,l \leq n$. The noncommutative Steinberg unitary algebra is a noncommutative analog of the Steinberg unitary Lie algebra defined in [2].

If all $\gamma_i = 1$, we let $eu_n(R,-)$ and $\mathfrak{stul}_n(R,-)$ denote $eu_n(R,-,\gamma)$ and $\mathfrak{stul}_n(R,-,\gamma)$, respectively.

Next, we will see that the noncommutative Steinberg unitary algebra is a generalization of the noncommutative Steinberg algebra.

Proposition 2.3. *Let S be an associative algebra with identity. Let $(R,-) = (S \oplus S^{op}, ex)$. Then*

$$\mathfrak{stl}_n(S) \cong \mathfrak{stul}_n(R,-).$$

Proof. First, there is a Leibniz algebra homomorphism

$$\xi : \mathfrak{stul}_n(R,-) \to \mathfrak{stl}_n(S)$$

such that $\xi(u_{ij}((a,b))) = X_{ij}(a) - X_{ji}(b)$, for all $a,b \in S$. It is obvious that ξ is onto. Also, $u_{ij}((a,0))$ satisfies the relations (2.1)–(2.3). It follows that there exists a Leibniz algebra homomorphism $\xi^- : \mathfrak{stl}_n(S) \to \mathfrak{stul}_n(R,-)$ such that $\xi^-(X_{ij}(a)) = u_{ij}((a,0))$. It is easy to see that $\xi^- \circ \xi(u_{ij}((a,b))) = u_{ij}((a,b))$ which shows that ξ is one - to - one. □

Remark 2.4. One can similarly show that $eu_n(R,-) \cong sl_n(S)$ as Lie algebras, see for example [2].

Setting

$$T_{ij}(a,b) = [u_{ij}(a), u_{ji}(b)], \tag{2.10}$$

for $a,b \in R, 1 \le i \ne j \le n$. One can check that

$$[T_{ij}(a,b), u_{ik}(c)] = -[u_{ik}(c), T_{ij}(a,b)] = u_{ik}((ab - \overline{ab})c). \tag{2.11}$$

for $a,b,c \in R$ and distinct i,j,k. Using (2.11), we obtain

$$[T_{ij}(a,b), u_{ij}(c)] = -[u_{ij}(c), T_{ij}(a,b)] = u_{ij}((ab - \overline{ab})c + c(ba - \overline{ba})). \tag{2.12}$$

The following proposition is obvious.

Proposition 2.5. *Let $\mathfrak{T} := \sum_{1\le i<j\le n}[u_{ij}(R), u_{ji}(R)]$. Then $\mathfrak{T}$ is a subalgebra of $\mathfrak{stul}_n(R,-,\gamma)$ containing the center $\mathfrak{Z}$ of $\mathfrak{stul}_n(R,-,\gamma)$ with $[\mathfrak{T}, u_{ij}(R)] = [u_{ij}(R), \mathfrak{T}] \subseteq u_{ij}(R)$. Moreover,*

$$\mathfrak{stul}_n(R,-,\gamma) = \mathfrak{T} \oplus \sum_{1\le i<j\le n} \oplus u_{ij}(R). \tag{2.13}$$

Clearly, one has a Leibniz algebra epimorphism

$$\phi : \mathfrak{stul}_n(R,-,\gamma) \to eu_n(R,-,\gamma), \tag{2.14}$$

such that $\phi(u_{ij}(a)) = e_{ij}(a) - \gamma_i\gamma_j^{-1}e_{ji}(\bar{a})$. ϕ restricted to $u_{ij}(R)$ maps $u_{ij}(R)$ to $\boldsymbol{\xi}_{ij}$ and is one to one, for $1 \le i < j \le n$. It then follows that $\ker\phi \subseteq \mathfrak{Z}$.

Next, we will characterize $\ker\phi$. To do this, we need to understand more about the subalgebra $\mathfrak{T}$ of $\mathfrak{stul}_n(R,-,\gamma)$. As Proposition 2.5 in [9], one has

Proposition 2.6. *For any $a,b,c \in R$, and distinct i,j,k, we have*

(i) $T_{ij}(a,b) = T_{ji}(\bar{a},\bar{b})$,

(ii) $T_{ij}(a,bc) = T_{ik}(ab,c) + T_{kj}(ca,b)$,
(iii) $T_{ij}(a,1) + T_{ji}(a,1) = 0$,
(iv) $T_{ij}(a,1) = 0$, *if* $a \in R_+$.

Proof. (i) is clear. (ii) follows from the Leibniz identity.
Taking $b = c = 1$ in (ii), we get

$$T_{ij}(a,1) = T_{ik}(a,1) + T_{kj}(a,1) \tag{2.15}$$

and exchanging j and k in (2.15), we have

$$T_{ik}(a,1) = T_{ij}(a,1) + T_{jk}(a,1). \tag{2.16}$$

Combining (2.15) and (2.16) gives us

$$T_{kj}(a,1) + T_{jk}(a,1) = 0 \tag{2.17}$$

which is (iii). (iv) follows from (i) and (iii). □

Let

$$t(a,b) = T_{1j}(a,b) - T_{1j}(ba,1), \tag{2.18}$$

then one easily sees that $t(a,b)$ does not depend on the choices of j.
Now, we can interpret Proposition 2.6 as follows.

Proposition 2.7. *For* $a,b,c \in R$, *the following identities hold*:
(i) $t(a,b) - t(\bar{a},\bar{b}) = 0$,
(ii) $t(ab,c) - t(a,bc) + t(ca,b) = 0$.

Proof. (ii) follows from Proposition 2.6(ii). Also, from Proposition 2.6(ii), we get

$$T_{ij}(a,b) = T_{ik}(ab,1) + T_{kj}(a,b),$$

$$T_{ij}(a,b) = T_{ik}(a,b) + T_{kj}(ba,1).$$

So

$$T_{kj}(a,b) - T_{kj}(ba,1) = T_{ik}(a,b) - T_{ik}(ab,1). \tag{2.19}$$

It follows that

$$\begin{aligned} t(a,b) &= T_{1j}(a,b) - T_{1j}(ba,1) = T_{i1}(a,b) - T_{i1}(ab,1) \\ &= T_{1i}(\bar{a},\bar{b}) - T_{1i}(\overline{ab},1) = T_{1i}(\bar{a},\bar{b}) - T_{1i}(\bar{b}\bar{a},1) = t(\bar{a},\bar{b}). \end{aligned}$$

So (i) holds true. □

Note that $T_{ij}(a,b)$ is k-bilinear, and so is $t(a,b)$. As Lemma 2.8 in [9], one can show that

Lemma 2.8. *Every element $x \in \mathfrak{T}$ can be written as*

$$x = \sum_i t(a_i, b_i) + \sum_{2\le j\le n} T_{1j}(c_j, 1),$$

where $a_i, b_i \in R$, $c_j \in R_-$ (i.e., $\bar{c}_j = -c_j$).

It is easy to see that

$$\phi(t(a,b)) = \phi(T_{1j}(a,b) - T_{1j}(1,ba)) = e_{11}((ab-ba) - (\overline{ab-ba})). \tag{2.20}$$

Then, similarly to the proofs of Lemma 2.30 and Theorem 2.33 in [9], we have

Proposition 2.9.

$$\ker\phi = \left\{ \sum_i t(a_i,b_i) \mid \sum_i \overline{a_ib_i - b_ia_i} = \sum_i a_ib_i - b_ia_i \right\} \cong H_1^-(R).$$

We know that

$$\phi : \mathfrak{stul}_n(R,-,\gamma) \to eu_n(R,-,\gamma) \tag{2.21}$$

is a central extension. Clearly, $\mathfrak{stul}_n(R,-,\gamma)$ is perfect, so is $eu_n(R,-,\gamma)$, for $n \ge 3$. Next, we will determine when ϕ yields the universal central extension.

Theorem 2.10. *Assume that $(R,-)$ is an associative algebra such that R is a free k-module. Then*

$$HL_2(\mathfrak{stul}_n(R,-,\gamma)) = \begin{cases} (0) & \text{if } n\ge 5,\\ \mathcal{N}(R,-) & \text{if } n=4,\\ (0) & \text{if } n=3 \text{ and } \frac{1}{3}\in k. \end{cases}$$

If $n \ge 5$, the theorem can be proved as in Theorem 2.37 of [9].

If $n=3$ and $\frac{1}{3} \in k$, the theorem can be proved as Theorem 5.18 of [3]. The only thing is to recall that ad$z$ defined as in (1.3) is a derivation.

If $n=4$, the theorem can be proved as Theorem 6.19 of [3]. We will treat this case with more details.

Recall that in $\mathcal{N}(R,-)$, we have

$$\ell\_(\bar{a}b,c) - \ell\_(a,\bar{b}c) + \ell\_(\bar{c}a,b) = 0, \tag{2.22}$$

$$\ell\_((c(ab-\overline{ab}) + (ba-\overline{ba})c), d) = 0, \tag{2.23}$$

$$\ell\_(d,(c(ab-\overline{ab}) + (ba-\overline{ba})c)) = 0. \tag{2.24}$$

Next, we collect some identities which can be easily derived from (2.22)–(2.24).

Lemma 2.11. *For $a,b,c,d \in R$, we have*

$$\ell\_(a,b) = \ell\_(a,\bar{b}), \quad \ell\_(a,b) = \ell\_(\bar{a},b), \tag{2.25}$$

$$\ell\_(ab,c) - \ell\_(a,\bar{b}c) + \ell\_(ac,b) = 0, \tag{2.26}$$

$$\ell\_(a,\bar{b}c) - \ell\_(\bar{b}a,c) - \ell\_(\bar{a}c,b) = 0, \tag{2.27}$$

$$\ell\_(c,(ba - \overline{ba})d) - \ell\_((ab - \overline{ab})c,d) = 0. \tag{2.28}$$

Proof. Taking $b = c = 1$ in (2.23) and (2.24) gives (2.25). Replacing a by $\bar{a}$ in (2.22) and using (2.25) we obtain (2.26). Eqs. (2.27) also follows from (2.22) and (2.25).

Since $ba - \overline{ba} \in R\_$, it follows from (2.26) that

$$\ell\_(c(ba - \overline{ba}),d) + \ell\_(c,(ba - \overline{ba})d) + \ell\_(cd, ba - \overline{ba}) = 0$$

and so

$$\ell\_(c(ba - \overline{ba}),d) + \ell\_(c,(ba - \overline{ba})d) = 0. \tag{2.29}$$

Now (2.28) follows from (2.24) and (2.29). □

Let

$$\mathscr{G} = \mathfrak{stul}_4(R,-,\gamma).$$

Our goal is to show that $HL_2(\mathscr{G}) = \mathscr{N}(R,-)$. One can easily see that $\mathscr{G}$ is $\mathbb{Z}_2^4$-graded Leibniz algebra such that $\deg(u_{ij}(a)) = \varepsilon_i + \varepsilon_j$, where $\varepsilon_i = (0,\ldots,1,\ldots,0)$ with 1 in the ith place. Moreover, by Proposition 2.5,

$$\mathscr{G} = \mathfrak{stul}_4(R,-,\gamma) = \mathscr{G}_0 \oplus \coprod_{1\le i<j\le 4} \mathscr{G}_{\varepsilon_i+\varepsilon_j},$$

where

$$\mathscr{G}_0 = \mathfrak{T} = \sum_{1\le i<j\le 4} [u_{ij}(R), u_{ji}(R)] \qquad \text{and} \qquad \mathscr{G}_{\varepsilon_i+\varepsilon_j} = u_{ij}(R).$$

We now define a bilinear bracket on the k-module

$$\hat{\mathscr{G}} = \mathscr{N}(R,-) \oplus \mathfrak{stul}_4(R,-,\gamma)$$

by

$$[\mathscr{N}(R,-),\hat{\mathscr{G}}] = [\hat{\mathscr{G}},\mathscr{N}(R,-)] = (0), \tag{2.30}$$

$$[x,y] = \text{ the product } [x,y] \text{ in } \mathscr{G} \text{ for } x \in \mathscr{G}_\alpha, y \in \mathscr{G}_\beta, \alpha+\beta \neq \varepsilon, \tag{2.31}$$

$$[u_{12}(a),u_{34}(b)] = \ell\_(a,b) = [u_{34}(a),u_{12}(b)], \tag{2.32}$$

$$[u_{24}(a),u_{13}(b)] = -\gamma_3^{-1}\gamma_2\ell\_(a,b) = [u_{13}(a),u_{24}(b)], \tag{2.33}$$

and

$$[u_{32}(a),u_{14}(b)] = -\ell\_(a,b) = [u_{14}(a),u_{32}(b)], \tag{2.34}$$

where $\varepsilon = \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4$. Then $\hat{\mathscr{G}}$ is a $\mathbb{Z}_2^4$-graded algebra with $\hat{\mathscr{G}}_0 = \mathscr{G}_0, \hat{\mathscr{G}}_{\varepsilon_i+\varepsilon_j} = \mathscr{G}_{\varepsilon_i+\varepsilon_j}$ and $\hat{\mathscr{G}}_\varepsilon = \mathscr{N}(R,-)$. Moreover, one can check that the following identities hold. These can be done case by case.

$$\begin{aligned}[u_{ij}(a),u_{kl}(b)] &= [u_{kl}(a),u_{ij}(b)] \\ &= -[u_{il}(a),u_{kj}(b)] = -\gamma_k\gamma_j^{-1}[u_{ik}(a),u_{jl}(b)],\end{aligned} \tag{2.35}$$

$$[u_{ij}(ab),u_{kl}(c)] = [u_{ij}(a),u_{kl}(\bar{b}c)] - [u_{ij}(ac),u_{kl}(b)], \tag{2.36}$$

$$[u_{ij}(a),u_{kl}(\bar{b}c)] - [u_{ij}(\bar{b}a),u_{kl}(c)] - [u_{ij}(\bar{a}c),u_{kl}(b)] = 0, \tag{2.37}$$

$$[u_{ij}((ab-\overline{ab})c + c(ba-\overline{ba})),u_{kl}(d)] = 0, \tag{2.38}$$

$$[u_{ij}(c),u_{kl}((ba-\overline{ba})d)] - [u_{ij}((ab-\overline{ab})c),u_{kl}(d)] = 0 \tag{2.39}$$

for all $a,b,c,d \in R$.

Proposition 2.12. *$\hat{\mathscr{G}}$ is a Leibniz algebra.*

Proof. Since $\mathscr{G}$ is a Leibniz algebra, to prove the Leibniz identity in $\hat{\mathscr{G}}$ it suffices to check $J(x,y,z) = [x,[y,z]]-[[x,y],z]+[[x,z],y] = 0$ for $\deg(x)+\deg(y)+\deg(z) = \varepsilon$. We can also assume that $\deg(x)$, $\deg(y)$ and $\deg(z)$ are not equal to ε. This leaves only two possibilities:

Case 1: $\deg(x) = \varepsilon_i+\varepsilon_j, \deg(y) = \varepsilon_k+\varepsilon_l$ for distinct i,j,k,l, and $\deg(z) = 0$

Case 2: $\deg(x) = \varepsilon_i+\varepsilon_j, \deg(y) = \varepsilon_i+\varepsilon_k, \deg(z) = \varepsilon_i+\varepsilon_l$ for distinct i,j,k,l.

For Case 1, we only check the following two subcases and omit the other cases since they are very similar.

When $x = u_{ij}(a), y = u_{kl}(d), z = T_{ij}(a,b)$, we have from (2.12) and (2.38),

$$\begin{aligned}J(x,y,z) &= [[u_{ij}(c),T_{ij}(a,b)],u_{kl}(d)] \\ &= [u_{ij}((ab-\overline{ab})c + c(ba-\overline{ba})),u_{kl}(d)] = 0.\end{aligned}$$

When $x = u_{ij}(a), y = u_{kl}(d), z = T_{ik}(a,b)$, we have from (2.11) and (2.39),

$$\begin{aligned}&[u_{ij}(c),[u_{kl}(d),T_{ik}(a,b)] + [[u_{ij}(c),T_{ik}(a,b)],u_{kl}(d)] \\ &\quad = [u_{ij}(c),u_{kl}((ba-\overline{ba})d)] - [u_{ij}((ab-\overline{ab})c),u_{kl}(d)] = 0.\end{aligned}$$

For Case 2, we suppose that $x = u_{ij}(a), y = u_{ik}(b), z = u_{il}(c)$. Then, using (2.35) and (2.37),

$$\begin{aligned}J(x,y,z) &= -\gamma_i\gamma_k^{-1}[u_{ij}(a),u_{kl}(\bar{b}c)] - \gamma_i\gamma_k^{-1}[u_{kj}(\bar{b}a),u_{il}(c)] - \gamma_i\gamma_j^{-1}[u_{jl}(\bar{a}c),u_{ik}(b)] \\ &= -\gamma_i\gamma_k^{-1}[u_{ij}(a),u_{kl}(\bar{b}c)] - \gamma_i\gamma_k^{-1}[u_{il}(\bar{b}a),u_{kj}(c)] - \gamma_i\gamma_j^{-1}[u_{ik}(\bar{a}c),u_{jl}(b)] \\ &= -\gamma_i\gamma_k^{-1}[u_{ij}(a),u_{kl}(\bar{b}c)] + \gamma_i\gamma_k^{-1}[u_{ij}(\bar{b}a),u_{kl}(c)] \\ &\quad -\gamma_i\gamma_j^{-1}(-\gamma_k^{-1}\gamma_j[u_{ij}(\bar{a}c),u_{kl}(b)])\end{aligned}$$

$$= -\gamma_i\gamma_k^{-1}([u_{ij}(a), u_{kl}(\bar{b}c)] - [u_{ij}(\bar{b}a), u_{kj}(c)] - [u_{ij}(\bar{a}c), u_{kl}(b)]$$
$$= 0,$$

which completes the proof. □

Define $\pi : \hat{\mathscr{G}} \to \mathscr{G}$ by $\pi(\mathscr{N}(R,-)) = (0)$ and $\pi|_{\mathscr{G}} = id$. Then, it follows that $(\hat{\mathscr{G}}, \pi)$ is a central extension of $\mathscr{G}$. We will show that $(\hat{\mathscr{G}}, \pi)$ is the universal central extension of $\mathscr{G}$. To do this, we define a Leibniz algebra $\mathscr{G}^\sharp$ to be the Leibniz algebra generated by the symbols $u^\sharp_{ij}(a)$, $a \in R$, $1 \le i \ne j \le 4$ and the k-module $\mathscr{N}(R,-)$, subject to the relations:

($1^\#$) $u^\sharp_{ij}(a) = u^\sharp_{ji}(-\gamma_i\gamma_j^{-1}\bar{a})$,

($2^\#$) $a \mapsto u^\sharp_{ij}(a)$ is a k-linear mapping,

($3^\#$) $[u^\sharp_{ij}(a), u^\sharp_{jk}(b)] = -[u^\sharp_{jk}(b), u^\sharp_{ij}(a)] = u^\sharp_{ik}(ab)$, for distinct i, j, k,

($4^\#$) $[\mathscr{N}(R,-), u^\sharp_{ij}(a)] = [u^\sharp_{ij}(a), \mathscr{N}(R,-)] = 0$, for distinct i, j,

($5^\#$) $[u^\sharp_{12}(a), u^\sharp_{34}(b)] = \ell\_(a,b) = [u^\sharp_{34}(a), u^\sharp_{12}(b)]$,

($6^\#$) $[u^\sharp_{24}(a), u^\sharp_{13}(b)] = -\gamma_3^{-1}\gamma_2\ell\_(a,b) = [u^\sharp_{13}(a), u^\sharp_{24}(b)]$,

($7^\#$) $[u^\sharp_{32}(a), u^\sharp_{14}(b)] = -\ell\_(a,b) = [u^\sharp_{14}(a), u^\sharp_{32}(b)]$,

where $a, b \in R$, $1 \le i, j, k \le 4$. Clearly, there is a unique Leibniz algebra homomorphism $\psi : \mathscr{G}^\sharp \to \hat{\mathscr{G}}$ such that $\psi(u^\sharp_{ij}(a)) = u_{ij}(a)$. Moreover, as Lemma 6.18 of [3], one can prove

Lemma 2.13. *$\psi : \mathscr{G}^\sharp \to \hat{\mathscr{G}}$ is an isomorphism.*

Now, we are in the position to prove the following result.

Proposition 2.14. *If $(R,-)$ is associative and free over k. Then $(\hat{\mathscr{G}}, \pi)$ is the universal central extension of $\mathfrak{stul}_4(R,-,\gamma)$ and hence $HL_2(\mathfrak{stul}_4(R,-,\gamma)) \cong \mathscr{N}(R,-)$.*

Proof. Suppose that

$$0 \longrightarrow \mathscr{V} \longrightarrow \tilde{\mathscr{G}} \xrightarrow{\tau} \mathfrak{stul}_4(R,-,\gamma) \longrightarrow 0$$

is a central extension of $\mathfrak{stul}_4(R,-,\gamma)$. We must show that there exists a Leibniz algebra homomorphism $\lambda : \hat{\mathscr{G}} \to \tilde{\mathscr{G}}$ so that $\tau \circ \lambda = \pi$. Thus, by Lemma 2.13, it suffices to show that there exists a Leibniz algebra homomorphism $\eta : \mathscr{G}^\sharp \to \tilde{\mathscr{G}}$ so that $\tau \circ \eta = \pi \circ \psi$.

Using a basis for R, we choose a preimage $\tilde{u}_{ij}(a)$ of $u_{ij}(a)$ under τ, $1 \le i \ne j \le 4$, $a \in R$, so that the elements $\tilde{u}_{ij}(a)$ satisfy the relations ($1^\#$) and ($2^\#$). For distinct i, j, k, let

$$[\tilde{u}_{ij}(a), \tilde{u}_{jk}(b)] = \tilde{u}_{ik}(ab) + v^j_{ik}(a,b)$$

where $v^j_{ik}(a,b) \in \mathscr{V}$. Take $l \notin \{i,j,k\}$. Then

$$[\tilde{u}_{li}(c), [\tilde{u}_{ij}(a), \tilde{u}_{jk}(b)]] = [\tilde{u}_{li}(c), \tilde{u}_{ik}(ab)],$$
$$[[\tilde{u}_{ij}(a), \tilde{u}_{jk}(b)], \tilde{u}_{li}(c)] = [\tilde{u}_{ik}(ab), \tilde{u}_{li}(c)].$$

By the Leibniz identity, we obtain

$$[\tilde{u}_{li}(c), \tilde{u}_{ik}(ab)] = [\tilde{u}_{lj}(ca), \tilde{u}_{jk}(b)] = -[\tilde{u}_{ik}(ab), \tilde{u}_{li}(c)]. \tag{2.40}$$

In particular, $[\tilde{u}_{li}(c), \tilde{u}_{ik}(b)] = [\tilde{u}_{lj}(c), \tilde{u}_{jk}(b)]$. It follows that $v^i_{lk}(c,b) = v^j_{lk}(c,b)$ which shows that v^i_{lk} is independent of the choice of i. Setting $v_{lk}(c,b) = v^i_{lk}(c,b)$, we have

$$[\tilde{u}_{li}(c), \tilde{u}_{ik}(b)] = \tilde{u}_{lk}(cb) + v_{lk}(c,b). \tag{2.41}$$

Taking $c = 1$, we have

$$[\tilde{u}_{li}(1), \tilde{u}_{ik}(b)] = \tilde{u}_{lk}(b) + v_{lk}(1,b). \tag{2.42}$$

Now for $l > k$, we replace $\tilde{u}_{lk}(b)$ by $\tilde{u}_{lk}(b) + v_{lk}(1,b)$, and rechoose $\tilde{u}_{kl}(b) = -\gamma_k\gamma_l^{-1}$ $\tilde{u}_{lk}(\bar{b})$. Then, the new elements $\tilde{u}_{ij}(b)$ still satisfy the relations ($1^\sharp$) and ($2^\sharp$). Moreover, we have for $l > k$ that

$$[\tilde{u}_{li}(1), \tilde{u}_{ik}(b)] = \tilde{u}_{lk}(b) = -[\tilde{u}_{ik}(b), \tilde{u}_{li}(1)]. \tag{2.43}$$

We next check that (2.43) holds for $l < k$. In fact, using (2.40) and $k > l$ we have

$$\begin{aligned}[\tilde{u}_{li}(1), \tilde{u}_{ik}(b)] &= [\tilde{u}_{lj}(b), \tilde{u}_{jk}(1)] \\ &= [-\gamma_l\gamma_j^{-1}\tilde{u}_{jl}(\bar{b}), -\gamma_j\gamma_k^{-1}\tilde{u}_{kj}(1)] = -\gamma_l\gamma_k^{-1}\tilde{u}_{kl}(\bar{b}) = \tilde{u}_{lk}(b).\end{aligned}$$

It follows from (2.40) and (2.43) that

$$[\tilde{u}_{lj}(a), \tilde{u}_{jk}(b)] = [\tilde{u}_{li}(1), \tilde{u}_{ik}(ab)] = \tilde{u}_{lk}(ab) = -[\tilde{u}_{jk}(b), \tilde{u}_{lj}(a)] \tag{2.44}$$

for $a, b \in R$ and distinct l, j, k. Thus, the elements $\tilde{u}_{ij}(a)$ satisfy ($3^\sharp$).

Next, for distinct i, j, k, l,

$$\begin{aligned}[\tilde{u}_{ij}(ab), \tilde{u}_{kl}(c)] &= [[\tilde{u}_{ik}(a), \tilde{u}_{kj}(b)], \tilde{u}_{kl}(c)] \\ &= -\gamma_k\gamma_j^{-1}[\tilde{u}_{ik}(a), \tilde{u}_{jl}(\bar{b}c)] + [\tilde{u}_{il}(ac), \tilde{u}_{kj}(b)].\end{aligned} \tag{2.45}$$

Meanwhile,

$$\begin{aligned}[\tilde{u}_{ij}(ab), \tilde{u}_{kl}(c)] &= [[\tilde{u}_{il}(a), \tilde{u}_{lj}(b)], \tilde{u}_{kl}(c)] \\ &= -[\tilde{u}_{il}(a), \tilde{u}_{kj}(cb)] + \gamma_k\gamma_j^{-1}[\tilde{u}_{ik}(a\bar{c}), \tilde{u}_{jl}(\bar{b})].\end{aligned} \tag{2.46}$$

Taking $b = c = 1$ in (2.45) and (2.46) and adding together gives us

$$2[\tilde{u}_{ij}(a), \tilde{u}_{kl}(1)] = 0. \tag{2.47}$$

But then taking $b = 1$ in (2.45), we get

$$[\tilde{u}_{ij}(a), \tilde{u}_{kl}(c)] = -\gamma_k\gamma_j^{-1}[\tilde{u}_{ik}(a), \tilde{u}_{jl}(c)], \tag{2.48}$$

while taking $b = 1$ in (2.46), we obtain

$$[\tilde{u}_{ij}(a), \tilde{u}_{kl}(c)] = -[\tilde{u}_{il}(a), \tilde{u}_{kj}(c)]. \tag{2.49}$$

It follows from (2.48) and (2.49) that (2.45) becomes

$$[\tilde{u}_{ij}(ab), \tilde{u}_{kl}(c)] = [\tilde{u}_{ij}(a), \tilde{u}_{kl}(\bar{b}c)] - [\tilde{u}_{ij}(ac), \tilde{u}_{kl}(b)]. \tag{2.50}$$

Also, interchange i and k, j and l in (2.48) respectively, we have

$$\begin{aligned}[\tilde{u}_{kl}(a), \tilde{u}_{ij}(c)] &= -\gamma_i\gamma_l^{-1}[\tilde{u}_{ki}(a), \tilde{u}_{lj}(c)] \\ &= -\gamma_k\gamma_j^{-1}[\tilde{u}_{ik}(\bar{a}), \tilde{u}_{jl}(\bar{c})] = [\tilde{u}_{ij}(\bar{a}), \tilde{u}_{kl}(\bar{c})].\end{aligned} \tag{2.51}$$

Finally, we have, using (2.12),

$$[\tilde{u}_{ij}((ab - \overline{ab})c + c(ba - \overline{ba})), \tilde{u}_{kl}(d)] = [[\widetilde{T}_{ij}(a,b), \tilde{u}_{ij}(c)], \tilde{u}_{kl}(d)] = 0 \tag{2.52}$$

and

$$[\tilde{u}_{kl}(d), \tilde{u}_{ij}((ab - \overline{ab})c + c(ba - \overline{ba}))] = [\tilde{u}_{kl}(d), [\widetilde{T}_{ij}(a,b), \tilde{u}_{ij}(c)]] = 0, \tag{2.53}$$

where $\widetilde{T}_{ij}(a,b) = [\tilde{u}_{ij}(a), \tilde{u}_{ji}(b)]$.

Taking $b = c = 1$ in (2.52) and (2.53) gives us

$$[\tilde{u}_{ij}(a - \bar{a}), \tilde{u}_{kl}(d)] = 0 \quad \text{and} \quad [\tilde{u}_{kl}(d), \tilde{u}_{ij}(a - \bar{a})] = 0, \tag{2.54}$$

hence (2.51) becomes

$$[\tilde{u}_{kl}(a), \tilde{u}_{ij}(b)] = [\tilde{u}_{ij}(a), \tilde{u}_{kl}(b)]. \tag{2.55}$$

Now, put

$$\tilde{\ell}\_(a,b) = [\tilde{u}_{12}(a), \tilde{u}_{34}(b)]$$

for $a, b \in \mathscr{A}$. Then we have

$$\begin{aligned}&\tilde{\ell}\_(ab,c) = \tilde{\ell}\_(a,\bar{b}c) - \tilde{\ell}\_(ac,b),\\ &\tilde{\ell}\_((ab - \overline{ab})c + c(ba - \overline{ba}), d) = 0,\\ &\tilde{\ell}\_(d, (ab - \overline{ab})c + c(ba - \overline{ba})) = 0\end{aligned}$$

for $a, b, c, d \in R$.

Thus, the elements $\tilde{u}_{ij}(a)$ and $\tilde{\ell}\_(a,b)$ satisfy the relations ($1^\sharp$)–($7^\sharp$), and so there exists a Leibniz algebra homomorphism $\eta : \mathscr{G}^\sharp \to \tilde{\mathscr{G}}$ so that $\eta(u_{ij}^\sharp(a)) = \tilde{u}_{ij}(a)$ for $1 \le i \ne j \le 4$ and $a \in R$. But then $\tau \circ \eta(u_{ij}^\sharp(a)) = \tau(u_{ij}(a)) = u_{ij}(a)$ and $\pi \circ \psi(u_{ij}^\sharp(a)) = \pi(u_{ij}(a)) = u_{ij}(a)$, and thus $\tau \circ \eta = \pi \circ \psi$ as required. □

Let $\mathscr{G} = \mathfrak{stul}_n(R, -, \gamma)$, then the associated Lie algebra $\mathscr{G}_{\text{Lie}} = \mathfrak{stu}_n(R, -, \gamma)$, the Steinberg unitary Lie algebra (see [3, 9] or [3]). Then we have

Proposition 2.15. *If* $(R,-)$ *is associative and free over* k. *Then*

$$HL_2(\mathfrak{stu}_n(R,-,\gamma)) = \begin{cases} \mathrm{Im}\,\mathfrak{B} & \text{if } n \geq 5, \\ \mathrm{Im}\,\mathfrak{B} \oplus \mathrm{Im}\,P & \text{if } n = 4, \\ \mathrm{Im}\,\mathfrak{B} & \text{if } n = 3 \text{ and } \frac{1}{3} \in k. \end{cases}$$

Now, consider $(R,-) = (S \oplus S^{\mathrm{ex}}, \mathrm{ex})$. Let $c = (1,-1) \in R$. Then $\bar{c} = -c$, c is invertible and lies in the center of R. Thus (2.23) says that $\ell\_(a,b) = 0$ for all $a,b \in R$. So $\mathcal{N}(R,-) = (0)$. Also, by Propositions 1.7, 2.3 and Remark 1.12, we get the following result which was due to Loday and Pirashvili [15] for $n \geq 5$.

Corollary 2.16. *If* S *is an associative algebra and free over* k. *Then* $(\mathfrak{stl}_n(S), \rho)$ *is the universal central extension of* $sl_n(S)$. *Moreover,* $HL_2(sl_n(S)) = H_1(S)$ *and* $HL_2(\mathfrak{st}_n(S)) = ImB$ *if* $n \geq 4$ *or if* $n = 3$ *and* $\frac{1}{3} \in k$.

Remark 2.17. If R is commutative, then we have the universal central extension

$$0 \to \Omega^1_{R|k} \to \mathfrak{stl}(R) \to sl(R) \to 0,$$

which is an exact sequence of Leibniz algebras, where $\mathfrak{stl}(R) = \mathfrak{stl}_\infty(R)$ and $sl(R) = sl_\infty(R)$. In the paragraph after (2.9) in [6], Bloch indicated the following exact sequence of k-modules:

$$0 \to \Omega^1_{R|k} \to t_{St}(R) \to sl(R) \to 0$$

and pointed out that $t_{St}(R)$ does not have a natural Lie algebra structure. However, it now becomes clear that $t_{St}(R) \cong \mathfrak{stl}(R)$ has a Leibniz algebra structure.

3. Analogues of noncommutative Steinberg unitary algebras of other types

In this section we assume that $\mathbb{Q} \subseteq k$, where $\mathbb{Q}$ is the rational field.

Let $\dot{\mathfrak{g}}$ be a finite dimensional split simple Lie algebra over k. It is well-known that $\dot{\mathfrak{g}}$ has a Chevalley basis $\{e_\alpha, h_\alpha, |\ \alpha \in \Delta\}$ (see [7] or [10]), where Δ is the root system of $\dot{\mathfrak{g}}$ with a base $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ and l is the rank of $\dot{\mathfrak{g}}$ (or Δ), satisfying $[h_\alpha, h_\beta] = 0$, $[h_\alpha, e_\beta] = \beta(h_\alpha)e_\beta$, $[e_\alpha, e_{-\alpha}] = h_\alpha$, $[e_\alpha, e_\beta] = N_{\alpha,\beta}e_{\alpha+\beta}$, where $\alpha, \beta \in \Delta$, if $\alpha + \beta \notin \Delta$, we set $N_{\alpha,\beta} = 0$.

One also has a Chevalley involution θ (see [7] or [10]), $\theta(e_\alpha) = -e_{-\alpha}$. Then one has (see [7]),

$$N_{\alpha,\beta} = -N_{\beta,\alpha} = -N_{-\alpha,-\beta} \tag{3.1}$$

for all $\alpha, \beta \in \Delta$.

Throughout this section, we will fix the Chevalley basis $\{e_\alpha, h_\alpha, |\ \alpha \in \Delta\}$ and the Chevalley involution θ chosen as above. Assume that R is an associative commutative k-algebra with identity, equipped with an involution $^-$.

In the k-Lie algebra $\dot{\mathfrak{g}} \otimes R$, we write $x(a) = x \otimes a$. Let $eu(\dot{\mathfrak{g}}; R, -)$ be the subalgebra of $\dot{\mathfrak{g}} \otimes R$ generated by the elements $e_\alpha(a) - e_{-\alpha}(\bar{a}), \alpha \in \Delta, a \in R$.

If $\alpha \pm \beta \neq 0$, then by using the Chevalley basis and (3.1),

$$[e_\alpha(a) - e_{-\alpha}(\bar{a}), e_\beta(b) - e_{-\beta}(\bar{b})] \\ = N_{\alpha,\beta}(e_{\alpha+\beta}(ab) - e_{-\alpha-\beta}(\bar{a}\bar{b})) + N_{-\alpha,\beta}(e_{\alpha-\beta}(a\bar{b}) - e_{-\alpha+\beta}(\bar{a}b)). \tag{3.2}$$

Also, it is easy to see that $eu(\dot{\mathfrak{g}}; R, -)$ has a k-module decomposition:

$$eu(\dot{\mathfrak{g}}; R, -) = \xi_0 \oplus \sum_{\alpha \in \Delta^+} \oplus \xi_\alpha, \tag{3.3}$$

where $\xi_\alpha = \{e_\alpha(a) - e_{-\alpha}(\bar{a}) | a \in R\}$, for $\alpha \in \Delta^+$, Δ^+ is the set of positive roots and ξ_0 is the subalgebra which is spanned by the elements

$$[e_\alpha(a) - e_{-\alpha}(\bar{a}), e_{-\alpha}(b) - e_\alpha(\bar{b})] = h_\alpha(ab - \bar{a}\bar{b}) \tag{3.4}$$

for $a, b \in R$.

For $l \geq 2$, $\mathfrak{stul}(\dot{\mathfrak{g}}; R, -)$ is defined to be the Leibniz algebra over k generated by the symbols $u_\alpha(a), \alpha \in \Delta, a \in R$, subject to the relations:

$$u_\alpha(a) = u_{-\alpha}(-\bar{a}), \tag{3.5}$$

$$a \mapsto u_\alpha(a) \text{ is a } k\text{-linear map}, \tag{3.6}$$

$$[u_\alpha(a), u_\beta(b)] = -[u_\beta(b), u_\alpha(a)] \\ = N_{\alpha,\beta} u_{\alpha+\beta}(ab) + N_{-\alpha,\beta} u_{\alpha-\beta}(a\bar{b}) \tag{3.7}$$

for $\alpha \pm \beta \neq 0, \alpha, \beta \in \Delta, a, b \in R$.

By analogy with the unitary case in Section 2 we call this the *noncommutative Steinberg unitary algebra* of $(\dot{\mathfrak{g}}, R, -)$.

Now let $H_\alpha(a, b) = [u_\alpha(a), u_{-\alpha}(b)]$. As Lemma 3.9 in [9], one can show that

Lemma 3.1. $[H_\alpha(a, b), u_\beta(c)] = -[u_\beta(c), H_\alpha(a, b)] = \beta(h_\alpha) u_\beta((ab - \overline{ab})c)$.

Clearly, we have a surjective homomorphism

$$\phi : \mathfrak{stul}(\dot{\mathfrak{g}}; R, -) \to eu(\dot{\mathfrak{g}}; R, -),$$

given by $\phi(u_\alpha(a)) = e_\alpha(a) - e_{-\alpha}(\bar{a})$.

Proposition 3.2. *$\mathfrak{H} := \sum_{\alpha \in \Delta^+} [u_\alpha(R), u_\alpha(R)]$ is a subalgebra of $\mathfrak{stul}(\dot{\mathfrak{g}}; R, -)$ with $[\mathfrak{H}, u_\alpha(R)] = [u_\alpha(R), \mathfrak{H}] \subseteq u_\alpha(R)$. Moreover,*

$$\mathfrak{stul}(\dot{\mathfrak{g}}; R, -) = \mathfrak{H} \oplus \sum_{\alpha \in \Delta^+} \oplus u_\alpha(R)$$

and $\ker \phi$ *is contained in the center of* $\mathfrak{stul}(\dot{\mathfrak{g}}; R, -)$.

As in Section 2, we further analyze the structure of $\mathfrak{H}$.

Proposition 3.3. *Let $\alpha, \beta \in \Delta$, suppose $\alpha + \beta \in \Delta$, then*

(i) $H_\alpha(a, b) = H_{-\alpha}(\bar{a}, \bar{b})$,

(ii) $N_{\alpha,\beta}H_{\alpha+\beta}(ab,c) = N_{\beta,-\alpha-\beta}H_\alpha(a,bc) + N_{\alpha,-\alpha-\beta}H_{-\beta}(ac,b)$,
(iii) $H_{-\beta}(a,1) + H_\beta(a,1) = 0$,
(iv) $H_\beta(c,1) = 0$, *for* $c \in R_+$
for $a,b,c \in R$.

Proof. (i) is clear. (ii) follows from the Leibniz identity.
Taking $b = c = 1$ in (ii), we have

$$N_{\alpha,\beta}H_{\alpha+\beta}(a,1) = N_{\beta,-\alpha-\beta}H_\alpha(a,1) + N_{\alpha,-\alpha-\beta}H_{-\beta}(a,1). \tag{3.8}$$

Note that $\alpha = (\alpha+\beta) + (-\beta)$. It follows from (3.8) that

$$N_{\alpha+\beta,-\beta}H_\alpha(a,1) = N_{-\beta,-\alpha}H_{\alpha+\beta}(a,1) + N_{\alpha+\beta,-\alpha}H_\beta(a,1). \tag{3.9}$$

Adding (3.8) and (3.9) together, we obtain

$$N_{\alpha,-\alpha-\beta}(H_{-\beta}(a,1) + H_\beta(a,1)) = 0$$

which yields (iii). It then follows that

$$H_\beta(\bar{a} + a,1) = 0, \tag{3.10}$$

which is (iv). □

Set $h^\alpha(a,b) = H_\alpha(a,b) - H_\alpha(ab,1)$.
If $\alpha, \beta, \alpha+\beta \in \Delta$, then by taking $b = 1$ in Proposition 3.3(ii), we have

$$N_{\alpha,\beta}H_{\alpha+\beta}(a,c) = N_{\beta,-\alpha-\beta}H_\alpha(a,c) + N_{\alpha,-\alpha-\beta}H_{-\beta}(ac,1). \tag{3.11}$$

Taking $c = 1$ in Proposition 3.3(ii), we get

$$N_{\alpha,\beta}H_{\alpha+\beta}(ab,1) = N_{\beta,-\alpha-\beta}H_\alpha(a,b) + N_{\alpha,-\alpha-\beta}H_{-\beta}(a,b). \tag{3.12}$$

Letting $b = c$ in (3.12), and comparing (3.11) with (3.12), we obtain

$$N_{\alpha,\beta}(H_{\alpha+\beta}(a,c) - H_{\alpha+\beta}(ac,1)) = -N_{\alpha,-\alpha-\beta}(H_{-\beta}(a,c) - H_{-\beta}(ac,1)). \tag{3.13}$$

Exchanging α and β in (3.13) we get

$$N_{\beta,\alpha}(H_{\alpha+\beta}(a,c) - H_{\alpha+\beta}(ac,1)) = -N_{\beta,-\alpha-\beta}(H_{-\alpha}(a,c) - H_{-\alpha}(ac,1)),$$

and so

$$N_{\alpha,-\alpha-\beta}h^{-\beta}(a,c) = -N_{\beta,-\alpha-\beta}h^{-\alpha}(a,c), \tag{3.14}$$

Replacing a by ab in (3.8), gives us

$$N_{\alpha,\beta}H_{\alpha+\beta}(ab,1) = N_{\beta,-\alpha-\beta}H_\alpha(ab,1) + N_{\alpha,-\alpha-\beta}H_{-\beta}(ab,1). \tag{3.15}$$

Comparing (3.12) and (3.15), we obtain

$$N_{\beta,-\alpha-\beta}h^\alpha(a,b) + N_{\alpha,-\alpha-\beta}h^{-\beta}(a,b) = 0. \tag{3.16}$$

From (3.14), (3.16) and (i), we have

$$h^{\alpha}(a,b) = h^{-\alpha}(a,b) = h^{\alpha}(\bar{a},\bar{b}). \tag{3.17}$$

Thus, we have

Proposition 3.4. *If $\beta \in \Delta$, then for $a,b,c \in R$, the following identities hold*:
(i) $h^{\beta}(a,b) - h^{\beta}(\bar{a},\bar{b}) = 0$,
(ii) $h^{\beta}(ab,c) - h^{\beta}(a,bc) + h^{\beta}(ca,b) = 0$.

Note that $H_{\alpha}(a,b)$ is k-bilinear, and so is $h^{\alpha}(a,b)$. As Lemma 2.8, one has

Lemma 3.5. *Every element $x \in \mathfrak{H}$ can be written as*

$$x = \sum_{i} h^{\alpha_1}(a_i,b_i) + \sum_{j=1}^{l} H_{\alpha_j}(1,c_j),$$

where $a_i, b_i \in R$, $c_j \in R_{-}$.

It is easy to see that $\phi(h^{\alpha_1}(a,b)) = 0$. Moreover, we have the following result whose proof is similar to the proofs of Proposition 3.34 and Theorem 3.35 in [9].

Proposition 3.6. $\ker \phi = \{\sum_i h^{\alpha_1}(a_i,b_i) \mid a_i, b_i \in R\} \cong H_1^{-}(R)$.

We know that $\phi : \mathfrak{stul}(\dot{\mathfrak{g}}; R, -) \to eu(\dot{\mathfrak{g}}; R, -)$ is a central extension. We will determine when ϕ yields the universal central extension. Towards the end, we set (see [9])

Assumption 3.7. There exists an element $e \in R$ such that $\bar{e} = -e$, and e is invertible.

With Assumption 3.7, we have, by using Lemma 3.1

$$\begin{aligned}[H_{\alpha}(e,1), H_{\beta}(e,1)] &= [H_{\alpha}(e,1), [u_{\beta}(e), u_{-\beta}(1)]] \\ &= [[H_{\alpha}(e,1), u_{\beta}(e)], u_{-\beta}(1)] - [[H_{\alpha}(e,1), u_{-\beta}(1)], u_{\beta}(e)] \\ &= [\beta(h_{\alpha})u_{\beta}(2e^2), u_{-\beta}(1)] + [\beta(h_{\alpha})u_{-\beta}(2e), u_{\beta}(e)] \\ &= 2\beta(h_{\alpha})H_{\beta}(e^2,1) + 2\beta(h_{\alpha})H_{-\beta}(e,e) \\ &= 2\beta(h_{\alpha})H_{-\beta}(e,e)\end{aligned} \tag{3.18}$$

as $e^2 \in R_{+}$ and $H_{\beta}(e^2,1) = 0$. Also, by Lemma 3.1 again, we have

$$[u_{\alpha}(a), H_{-\beta}(e,e)] = 0 \tag{3.19}$$

for all $a \in R$.

Now, we can prove the following theorem:

Theorem 3.8. *If* $(R,-)$ *is an associative commutative algebra satisfying Assumption* 3.7 *and* R *is a free* k*-module, then* $HL_2(\mathfrak{stul}(\dot{\mathfrak{g}};R,-))=(0)$.

Proof. Suppose that the following is a central extension,

$$0\longrightarrow V\longrightarrow L\overset{\pi}{\longrightarrow}\mathfrak{stul}(\dot{\mathfrak{g}};R,-)\longrightarrow 0.$$

Let $\tilde{u}_\alpha(a)$ be the preimage of $u_\alpha(a)$ in L under π chosen as follows. It suffices to choose $\tilde{u}_\alpha(a)$ for $\alpha\in\Delta^+$ and $a\in\{r_\lambda\}_{\lambda\in\Lambda}$ which is a basis of R, and then extend our choices to all $a\in R$ and $\alpha\in\Delta^+$ by linearity and (3.5).

Let $\tilde{H}_\alpha(a,b)=[\tilde{u}_\alpha(a),\tilde{u}_{-\alpha}(b)]$. Recall that e is fixed in Assumption 3.7, then by Lemma 3.1, we have

$$[\tilde{u}_\alpha(e^{-1}a),\tilde{H}_\alpha(e,1)]=-4\tilde{u}_\alpha(a)+v_\alpha(a),$$

for some $v_\alpha(a)\in V$. Replacing $\tilde{u}_\alpha(a)$ by $\tilde{u}_\alpha(a)-v_\alpha(a)/4$, for $\alpha\in\Delta^+$, then using (3.5) get $\tilde{u}_\alpha(a)$ for $\alpha\in-\Delta^+$. So we get for $a\in R$,

$$[\tilde{u}_\alpha(e^{-1}a),\tilde{H}_\alpha(e,1)]=-4\tilde{u}_\alpha(a). \tag{3.20}$$

Now, we claim

$$[\tilde{u}_\alpha(a),\tilde{H}_{-\beta}(e,e)]=0. \tag{3.21}$$

Indeed, by (3.19), we have

$$\begin{aligned}0&=[[\tilde{u}_\alpha(e^{-1}a),\tilde{H}_{-\beta}(e,e)],\tilde{H}_\alpha(e,1)]\\&=[\tilde{u}_\alpha(e^{-1}a),[\tilde{H}_{-\beta}(e,e),\tilde{H}_\alpha(e,1)]]+[[\tilde{u}_\alpha(e^{-1}a),\tilde{H}_\alpha(e,1)],\tilde{H}_{-\beta}(e,e)]\\&=-4[\tilde{u}_\alpha(a),\tilde{H}_{-\beta}(e,e)],\end{aligned}$$

as $[\tilde{H}_{-\beta}(e,e),\tilde{H}_\alpha(e,1)]\in V$.

From (3.20), we have

$$[[\tilde{u}_\alpha(e^{-1}a),\tilde{H}_\alpha(e,1)],\tilde{H}_\beta(e,1)]=-4[\tilde{u}_\alpha(a),\tilde{H}_\beta(e,1)].$$

By the Leibniz identity, (3.18) and (3.21), the left-hand side is

$$\begin{aligned}&[\tilde{u}_\alpha(e^{-1}a),[\tilde{H}_\alpha(e,1),\tilde{H}_\beta(e,1)]]+[[\tilde{u}_\alpha(e^{-1}a)],\tilde{H}_\beta(e,1)],\tilde{H}_\alpha(e,1)]\\&\quad=[\tilde{u}_\alpha(e^{-1}a),2\beta(h_\alpha)\tilde{H}_{-\beta}(e,e)]+[-\alpha(h_\beta)\tilde{u}_\alpha(2a),\tilde{H}_\alpha(e,1)]=8\alpha(h_\beta)\tilde{u}_\alpha(ea).\end{aligned}$$

So we have

$$[\tilde{u}_\alpha(a),\tilde{H}_\beta(e,1)]=-2\alpha(h_\beta)\tilde{u}_\alpha(ea). \tag{3.22}$$

Note that $\tilde{u}_\alpha(a)$ satisfies the relations (3.5) and (3.6). As in the proof of Theorem 3.45 of [9], using (3.22), one can show that $\tilde{u}_\alpha(a)$ satisfies the relation (3.7). The proof is thus complete. □

Let S be an associative commutative k-algebra with identity. Suppose that (-1) has no square root in k. Define $R = S \otimes_k k(\mathrm{i}) = S \oplus \mathrm{i}S$ with involution $\overline{a+\mathrm{i}b} = a - \mathrm{i}b$, where $\mathrm{i} = \sqrt{-1}$, then R is a k-algebra equipped an involution $^-$. So $eu(\dot{\mathfrak{g}}; R, -)$ is a k-Lie algebra generated by

$$e_\alpha(a) - e_{-\alpha}(\bar{a}) = (e_\alpha - e_{-\alpha}) \otimes a,$$
$$e_\alpha(ia) + e_{-\alpha}(ia) = i(e_\alpha + e_{-\alpha}) \otimes a$$

for all $a \in S$, $\alpha \in \Delta$. Evidently, $eu(\dot{\mathfrak{g}}; R, -) = \dot{\mathfrak{g}}_c \otimes_k S$ where $\dot{\mathfrak{g}}_c$ is a k-Lie algebra generated by $\mathrm{e}_\alpha - \mathrm{e}_{-\alpha}$ and $i(\mathrm{e}_\alpha + \mathrm{e}^{-\alpha})$. If $k = \mathbb{R}$, the field of real numbers, then $\dot{\mathfrak{g}}_c$ is nothing but the compact form of $\dot{\mathfrak{g}}$. Note that $\mathrm{e} = \sqrt{-1}$ satisfies Assumption 3.7.

Corollary 3.9. $HL_2(\dot{\mathfrak{g}}_c \otimes_k S) = \Omega^1_{S|k}$.

Proof. The rank l (of $\dot{\mathfrak{g}}$) ≥ 2 case follows from Theorem 3.8 and Proposition 1.8 while the rank $l = 1$ case follows from Lemma 2.1, Proposition 2.9 and Theorem 2.10. □

Next, let S be an associative commutative algebra over k and take $(R, -) = (S \oplus S^{op}, \mathrm{ex})$, one has $eu(\dot{\mathfrak{g}}; R, -) \cong \dot{\mathfrak{g}} \otimes S$. Note that $e = (1, -1) \in R$ satisfies Assumption 3.7.

Corollary 3.10. $HL_2(\dot{\mathfrak{g}} \otimes_k S) \cong \Omega^1_{S|k}$.

Proof. The rank $l \geq 2$ case follows from Theorem 3.8 and Proposition 1.7 while the rank $l = 1$ case follows from Lemma 2.2, Proposition 2.9 and Theorem 2.10. □

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Roughly speaking the Lie algebra homology is related to the appearance of cyclic homology (as it is manifest in the original work of Tsygan and then of Loday-Quillen). Jean-Louis Loday, Daniel Quillen, Cyclic homology and the Lie algebra homology of matrices, Comm. Math. Helv. 59, n. 1, 565-591 (1984), doi. Lie algebra homology involves the Chevalley-Eilenberg chain complex, which in turns involves the exterior powers of the Lie algebra.Â In fact this new complex for Leibniz homology further generalizes to the case of Leibniz algebras, where it computes certain Tor groups for corepresentations of Leibniz algebras. Definition. Given a commutative unital ring.