Regularity theory of Fourier integral operators with complex phases and singularities of affine fibrations

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Summary. The main subject of the present monograph is the regularity theory of Fourier integral operators with real and complex phases and related questions of the singularity theory of affine fibrations. It is known that the geometry and singularities of the canonical relation of an operator are reflected in the boundedness properties of operators in L^p-spaces. Therefore, a part of the book is devoted to the general singularity theory of affine fibrations. Conditions for the boundedness of Fourier integral operators are related to the singularities of affine fibrations in (subsets of) $\mathbb{C}^n$, defined by the Hessians of phase functions.

One of the new aspects of the current work is that we deal with operators with complex phase functions. The theory of such operators is well developed but their regularity has not been much studied. In a way, the use of complex phases provides a more natural approach to Fourier integral operators. In the monograph the so-called smooth factorization condition is extended to to the complex phases and a number of regularity properties are derived in the complex setting.

The use of the complex phase allows to treat several new examples, such as non-hyperbolic Cauchy problems for pseudo-differential equations and the oblique derivative problem. The background information on Fourier integral operators with real and complex phases as well as the singularity theory of affine fibrations and relevant methods of complex analytic geometry is provided in the book.

Contents

Introduction

The present monograph deals with two different but related topics. The first one is the regularity theory of Fourier integral operators. We restrict our attention to non-degenerate operators but allow phase functions to be complex valued. The non-degeneracy means that the canonical relation of the operator is locally a graph of a diffeomorphism between cotangent bundles of two manifolds. Such operators arise naturally as propagators (solution operators) for partial differential equations. The theory of these operators is well developed and we refer the reader to excellent monographs [11], [27], [75]. Regularity properties of these operators have been under study for a long time. The L^2 boundedness of operators of zero order was established in [25], but was essentially known in different forms before (see, for example, [16]). The boundedness properties of Fourier integral operators in L^p spaces for $1 < p < \infty$ were established in [63]. The authors prove that operators of order $-(n-1)|1/p - 1/2|$ are bounded from $L^p_{comp}(Y)$ to $L^p_{loc}(X)$, where $n = \dim X = \dim Y$. However, in this paper the authors came up with an interesting condition called "the smooth factorization condition", which allows to improve L^p estimates given certain information on the dimension of the singular support of the Schwartz integral kernel of the operator and singularities of its wave front. All this concerns operators with real phase functions.

Fourier integral operators with complex phases appear naturally in different problems. The theory of operators with complex phases was systematically developed in [37] and [38]. It was used in [75] to describe solution operators for the Cauchy problem for partial differential operators with complex characteristics. L^2 boundedness of these Fourier integral operators of zero orders is due to [38], and to [26] in greater generality. The complex phase is used in the analysis of the oblique derivative problems ([38]) and in the description of projections to kernels and cokernels of pseudo-differential operators with non-involutive characteristics ([12]). In a way the use of complex phase functions is more natural than the real ones. First, there are no geometric obstructions like the non-triviality of Maslov cohomology class. Second, one can use a single complex phase function to give a global parameterization of a Fourier integral operator ([33], or [62]).

The first chapter of this monograph describes the L^p properties of Fourier integral operators with complex phases. We suggest a local graph type condition (L) which insures that operators of order $-(n-1)|1/p - 1/2|$ with complex phases are bounded from L^p_{comp} to L^p_{loc}. If the imaginary part of the phase function is zero, that is if the phase function is real, our condition (L) is equivalent to the local graph condition for real phase functions. In Section 1.12 we propose the smooth factorization type condition (F) for the complex phase. Again, if the phase function is real valued, our condition (F) is equivalent to the smooth factorization condition of [63]. Under this condition we establish better L^p properties of operators. We provide the reader with the necessary background information on Fourier integral operators in Sections 1.1, 1.2, 1.3, as well as the overview of the regularity theory for real and complex phase functions in Sections 1.4 and 1.6.

It is convenient to use the complex domain for the analysis of the smooth factorization condition. This is the second topic of the present monograph. We will state a general problem of singularities of affine fibrations which includes the smooth factorization condition as a particular case. Let Ω be an open subset of $\mathbb{C}^n$. An affine fibration in Ω is a family of affine subspaces of Ω which locally do not intersect and whose union equals to almost the whole of Ω. These subspaces will be controlled by kernels of holomorphic mappings. To be more precise, let $A : \Omega \to \mathbb{C}^{p\times n}$ be a holomorphic matrix valued mapping. Let $k = \max_{\xi\in\Omega} \operatorname{rank} A(\xi)$ be the maximal rank of A in Ω. The set $\Omega^{(k)}$ where it is maximal, is open and dense in Ω, and on this set the mapping

$$\varkappa : \xi \mapsto \ker A(\xi)$$

is regular from $\Omega^{(k)}$ to the space of all $(n-k)$-dimensional linear subspaces of Ω. Our main condition will be that $\varkappa$ defines a fibration in $\Omega^{(k)}$, which means that for $\xi \in \Omega^{(k)}$, $\varkappa$ is constant on $\xi + \ker A(\xi)$. This setting will be made more precise as conditions (A1), (A2) in Chapter 2.

An important case occurs when $A = D\Gamma$ is the Jacobian of a holomorphic mapping $\Gamma : \Omega \to \mathbb{C}^p$. In this case Γ is constant on $\xi + \varkappa(\xi)$ for all $\xi \in \Omega^{(k)}$. This means that the level set $\Gamma^{-1}(\Gamma(\xi))$ is an affine subspace of Ω, equal to $\xi + \varkappa(\xi)$. Such fibrations will be called Jacobian. Conditions (A1), (A2) will be called (ΓA1), (ΓA2) in this case. If Γ itself is a gradient of a holomorphic mapping $\phi : \Omega \to \mathbb{C}$, the fibration will be called a gradient fibration. In this case $A(\xi) = D^2\phi(\xi)$ is the Hessian of ϕ. In our applications ϕ will be the complex analytic extension of the phase function of a Fourier integral operator when the phase function is real analytic.

In Chapter 2 we will study these fibrations and especially their singular sets. It turns out that the mapping $\varkappa$ is meromorphic and we can use methods of complex analytic geometry. All the background information will be provided in Section 2.5.

Chapter 3 is complementary to Chapter 2 and there we will study fibrations of gradient type in both real and complex setting. In Chapter 4 we will apply results of Chapters 1 and 2 to derive further estimates for analytic Fourier integral operators. Those are operators whose phase function is real analytic. Based on the estimates for the set of singularities for corresponding fibrations in Chapter 2, we will show that the smooth factorization type condition (F) holds automatically in a number of cases.

Finally, in Chapter 5 the analysis will be applied to several problems. In Section 5.1 we will describe several applications of the regularity theory of Fourier integral operators. We will go on to discuss regularity properties of solutions of hyperbolic equations in Section 5.2. In Section 5.4 we will allow the characteristic roots of a partial pseudo-differential equation to be complex. However, in order to be able to apply the theory of Fourier integral operators with complex phases, we will assume that the imaginary part of characteristic roots is non-negative. Then, according to [75], the Cauchy problem is well posed and its propagator is a Fourier integral operator with complex phase. We will derive estimates for fixed-time solutions of these equations in L^p spaces. We will also discuss the improvements when the smooth factorization type condition (F) is satisfied. In some cases it is satisfied automatically, for example in $\mathbb{R}^4$ or $\mathbb{R}^5$, when coefficients of the operator may depend on time, but not on the other variables. As another application, we will briefly discuss L^p estimates for the oblique derivative problem in Section 5.5.

The present monograph is based on the author's doctoral thesis. However, there only operators with real phases were investigated, meanwhile the emphasis of this monograph is on operators with complex phases. Parts of this book have appeared in several papers. Chapter 4 is a complex valued phase version of [52], where analytic operators with real phases were considered. Some results of Section 5.2 have appeared in [53] and in [55]. Section 1.11 has appeared in [54]. A survey of the regularity theory of operators with real phases has appeared in [56]. However, Chapter 2 presents a more general problem (A1), (A2) for holomorphic matrix valued functions. Some of its results were announced in [57] and [58]. We would like to mention the paper [59], which is related to the L^p estimates under the failure of the factorization condition. However, we did not feel it would take an integral part in this book and we mention it only briefly in Remark 1.12.4. The results on the complex phase were announced in [60]. Above all, in this book we tried to emphasize the geometric role played by the affine fibrations in the regularity theory, especially in the case of complex phase functions in Section 1.12.

Finally, it is a pleasure for me to thank several people who have contributed in one way or another to the appearance of this work. First, I would like to thank Hans Duistermaat for all the support which I have had from him during my years at the Utrecht University as his graduate student. His influence on my understanding of Fourier integral operators and my work can not be overestimated. I would like to thank departments of mathematics of Utrecht University, the Johns Hopkins University and University of Edinburgh, and finally the Imperial College, where I was able to continue to work on this project. I am also grateful to Chris Sogge, Andreas Seeger, Anders Melin, Ari Laptev and Yuri Safarov for interesting discussions about real and complex phases.

We study Fourier integral operators of HÃ¶rmander's type acting on the spaces $\mathcal F L^p(\mathbb R^d)_{\rm comp} $, $1\leq p\leq\infty$, of compactly supported distributions whose Fourier transform is in L^p. We show that the sharp loss of derivati.Â We study Fourier integral operators of HÃ¶rmander's type acting on the spaces $\mathcal F L^p(\mathbb R^d)_{\rm comp} $, $1\leq p\leq\infty$, of compactly supported distributions whose Fourier transform is in L^p. We show that the sharp loss of derivatives for such an operator to be bounded on these spaces is related to the rank r of. the Hessian of the phase ${\mit\Phi}(x,\eta)$ with respect to the space variables x. The Fourier transform is just the integral part of this. What that means is that instead of looking at the center of mass, you would scale it up by some amount. If the portion of the original graph you were using spanned three seconds, you would multiply the center of mass by three.Â Even though in practice, with things like sound editing, you'll be integrating over a finite time interval, the theory of Fourier transforms is often phrased where the bounds of this integral are -âˆž and âˆž. Concretely, what that means is that you consider this expression for all possible finite time intervals, and you just ask, "What is its limit as that time. Abstract: We consider regularity properties of Fourier integral operators in various function spaces. The most interesting case is the L^p spaces, for which survey of recent results is given. For example, sharp orders are known for operators satisfying the so-called smooth factorization condition. Here this condition is analyzed in both real and complex settings.Â Singularities of such fibrations are analyzed in this paper in the general case. In particular, it is shown that if the dimension n or the rank of the Jacobi matrix is small, then all singularities of an affine fibration are removable. The fibration associated with a Fourier integral operator is given by the kernels of the Hessian of the phase function of the operator.