Stochastic Differential Equations with Jumps[1]

Jose-Luis Menaldi[2]

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[2] Wayne State University, Department of Mathematics, Detroit, MI 48202, USA (e-mail: jlm@math.wayne.edu).

[3] This book is being progressively updated and expanded. If you discover any errors or you have suggested improvements please e-mail the author.

[4] TEMPORARY: All sections in RED need some revision, specially Part III

Rationality for the book

In *Deterministic Control*, depending on how the time is regarded as either continuous or discrete, two models can be set, which combined yields the so called hybrid system. The state representation of the continuous model evolves following an ordinary differential equation (ODE) of the form

$$\dot{x}(t) = A(t)x(t) + B(t)v(t), \tag{1}$$

where $t \geq 0$ is the time, $x = x(t)$ is the state and $v = v(t)$ is the control. The state x (in $\mathbb{R}^n$) represents all variables needed to describe the physical system and the control v (in $\mathbb{R}^m$) contains all parameters that can be modified as time passes. The matrices $A(t)$ and $B(t)$ are the coefficients of the system.

The first question one may ask is the validity of the model, which lead to the *identification* of the coefficients. Next, one may want to *control* the system, i.e., to start with an initial state $x(t_0) = x_0$ and to *drive* the system to a prescribed position $x(t_1) = x_0$. Variations of this question are well known and called *controllability*.

Furthermore, another equation appear,

$$y(t) = C(t)x(t), \tag{2}$$

where $y = y(t)$ is the observation of the state and $C(t)$ is another coefficient. Clearly, y is in $\mathbb{R}^d$ with $d \leq n$. Thus, the problem is to reconstruct the state $\{x(t) : t_0 \leq t \leq t_1\}$ based on the observations $\{y(t) : t_0 \leq t \leq t_1\}$, which is called *observability*.

Another key question is the *stabilization* of the system, where one looks for a feedback, i.e., $v(t) = K(t)y(t)$ such that the closed system of ODE (1) and (2) is stable.

Variation of theses four basic questions: identification, controllability, observability and stabilization are solved in text books.

To each control (and state and observation) a cost (or profit) is associated with the intention of being minimized (or maximized), i.e., a performance index of the form

$$J = \int_0^T [y(t)]^* R(t)y(t)\mathrm{d}t + \int_0^T [v(t)]^* N(t)v(t)\mathrm{d}t \tag{3}$$

is to be optimized. This is called *optimal control*.

Two methods are available to solve optimal control problems, namely, the Pontryagin maximum principle and the Bellman dynamic programming. The above (1), (2), (3) linear-quadratic model can be successfully solved by either method. The maximum principle transforms the given (infinite-dimensional optimization) problem into ODE with initial and terminal conditions and a finite-dimensional optimization problem, i.e., a Lagrange multiplier technique. The dynamic programming transforms the given problem into a non-linear partial differential equation (PDE). There is a vast bibliography under the subject *optimal control*, e.g. classic references such as the text book Bertsekas [19], and Fleming and Rishel [74] or more recently Bardi and Capuzzo-Dolcetta [8], among others.

The ODE defining the evolution equations (of the state and the observation) may be nonlinear and the performance index may have a more general form. Moreover, the state could be distribute, i.e., the evolution equation becomes a PDE. Again, there are many references on the subject.

Both, the maximum principle and the dynamic programming are innovations over the classic calculus of variations. The positive part of the maximum principle is the preservation of the equation type (i.e., if the evolution equation is an ODE then the maximum principle equation is an ODE), and the negative part is the open-loop solution (i.e., the optimal control is of the form $v = v(t)$). On the other hand, the positive part of the dynamic programming is the closed-loop of feedback control (i.e., the optimal control has the form $v = K(t, x(t))$), while the negative part is the new equation (i.e., if the evolution equation is an ODE then the dynamic programming equation is an PDE). It is clear that this material is built on the ODE theory.

In *Stochastic Control*, an *uncertainty* component is added to the previous model. The coefficients becomes *random* and the evolution equation includes a *noise*. Perhaps the most typical example is presented in *signal processing*, where the signal (say x) has some noise. The ODE becomes stochastic

$$\dot{x}(t) = g(t, x(t), v(t)) + (\text{noise}). \tag{4}$$

Since Gauss and Poisson distributions are the main examples of continuous and discrete distributions, the driving noise is usually a Wiener process or a Poisson measure. Again, the four basic questions are discussed. Observability becomes *filtering*, which is very importance. Perhaps the most practical situation is the case with a linear state space and linear observation, which produces the celebrated Kalman filter. Clearly, an average performance index is used for the optimal stochastic control. Again, there is a vast bibliography on stochastic control from variety of points of view, e.g., Fleming and Soner [75], Yong and Zhou [242], Zabczyk [244], among others.

It is clear that stochastic control is mainly based on the theory of stochastic differential equations, which is the main subject of this book.

A Short History

This book-project was developed over a period of many years, first I wanted to collect a number of useful facts about diffusion with jumps, but in the doing so, other ideas appear and I decided to write the material in a lecture notes form, with more details, mainly for my own use. Part of this material was used to deliver some graduate course here and there, but, little by little the project becomes too big or too long for me.

With time, several improvement were made, and searching for possible collaborators, first in my mind and latter in the real world, I was disappointed with my first choice, but very please later on. In particular, with the help of Prof. Luciano Tubaro, a revision of several points were successfully implemented, but, instead of culminated with a regular book published (with Birkhäuser, where we were in the process of signing the book contract) several years ago, more attention was given to the research papers in process and this book becomes inactive, again. Recently, I reshaped and shorted a little the content to submit it to the American Mathematical Society, and about a year later, a negative report (with no specific points discussed) was sent to me, staying in general lines, that the book should be rewritten and it is of no use as it is. Naturally, I was not happy with the decision, but the reviewer was certainly, an important mathematician specialized in probability. After recovering from this fact, I sent a revised version to another nonprofit publisher (Princeton Press) and after almost another year, while inquiring for a report on my book, I was told how tricky is to market books nowadays and I was asked for specific features that would make this book marketable.

At this point, perhaps I should mention my experience with my two previous books (with Prof. M.G. Garroni), where we dedicated a continuous effort of more than 7 years (although not exclusively!) to write about 600 pages on the Green function and having about 5% of the earning (to be shared with my co-author) of sell profits, essentially to libraries in the world. Thus, with all this short history, I intent to justify my decision of making this internet published book.

Thanks

I would like to express my sincere thanks to Luciano Tubaro for his help in several portions of this book, and to several colleagues (a long undisclosed list goes here) for their comments on certain part and versions of this "unfinished" book. Last, but not less, I owe a great deal of gratitude to my wife, Cristina, who gave me moral support during the many years of preparation and realization of this book.

Preface

The purpose of this book is to develop a solid foundation on Stochastic Differential Equations so that Stochastic Optimal Control can be widely treated. A solid course in measure theory and Lebesgue spaces is a prerequisite, while some basic knowledge in functional spaces and probability is desired. The beginner will find several entry levels to the text, while the expert may access any Chapter (or Section) with minimum reference to previous material. Each Chapter has been kept as independent as possible.

A quick look at the contents will convince the reader that actually, each chapter could be developed into a full lecture notes. However our choice is to comfortably place ourselves within probability (stochastic processes) and (functional) analysis, to be able to properly describe the mathematical aspect of the stochastic differential equations needed for the dynamic programming technique, specifically, for controlled diffusion processes with jumps in a d-dimensional region. In a way, this material may be called *Complements of Stochastic Analysis* which could be used as a neutral introduction to stochastic optimal control, without any direct application, only the state of the system is considered.

Starting at a good elementary level of measure theory, the first and second chapters are an introduction to Markov-Feller processes, in the form of an extended summary of what may be necessary to consult (to follow our main subject). Thus, Chapter 1 is essentially addressed to the analyst with little background in probability while Chapter 2 presents an overview of semigroups. Only precise references to proofs are given and therefore these two first chapters are hard to read for the first time. Moreover, they may be considered as service chapters, where notation and fundamental concepts are briefly discussed.

Our main material begins in Chapter 3, where we study continuous and right-continuous with left-hand limits stochastic processes, locally bounded variation processes, martingales, piecewise deterministic processes and Lévy processes. Chapter 4 treats Gaussian processes, random measures, stochastic integrals and differentials. Chapter 5 is dedicated to the construction of d-dimensional (controlled) diffusion processes with jumps, by means of stochastic ordinary differential equations and martingale problems. Basic proofs are kept to be a minimum while focussing on estimates and key concepts. Each of these first five chapters is rather independent, in the sense that the material covered in each one can be considered as a *step further* into the *neighborhood* of Markov processes.

The reader with a good knowledge in Markov processes should skip the first three chapters, which should be taken as a short guided tour in stochastic analysis. Moreover, Chapter 3 can be viewed as a complement to the theory of processes with continuous sample paths, where more emphasis is given to Markov jump processes. Since some ideas of the main proofs are included, Chapter 3 is not so informal as the first two chapters. In Chapter 4, most of the details are given, the center is stochastic integration and differentials, but the focus is on estimates. This Chapter 4 can be skipped too, in a first reading, and coming back to it later if needed.

Someone familiar with Wiener processes and Itô stochastic integration will begin reading Chapter 4, and occasional go back to Chapter 3 to clear up some points in the discussion, and eventually to the end of Chapter 2, where second order integro-differential operators are briefly treated. Chapters 5 starts with stochastic differential equations, to end with the diffusion processes with jumps, as the main example of Markov-Feller processes. This is a formal chapter, the style theorems and proofs is used, and again, the focus is on estimates.

Next, Chapters 6 and 7 discuss advanced topics in stochastic differential equations, e.g., oblique boundary conditions, backward equations.

In a way, the introduction to each chapter is essentially a delimiter (on the style) to test the reader's background necessary to understand what follows. As part of a continuation, a second volume will be developed, which will include advanced topics in stochastic differential equations, e.g., oblique boundary conditions, backward equations and other more analytic tools. To complement the theory, some exercises are found only in Chapter 1. Other chapters are more lecture style, where the main points are carefully treated, but the instructor (or reader) should complement the theory.

Depending on the background of the reader, this book begins in Chapter 1 (hard to start there, but possible), or in Chapter 3 (adequate level in stochastic processes), or in Chapter 4 (good level in Markov processes), or in Chapter 5 (good level in martingales), or in Chapter 6 (advanced level in stochastic differential equations). Eventually, the reader should go back to previous Chapters (and the references mentioned there to check the proofs of some basic results) and keep on the side the analysis view point of Chapter 2. Essentially, each of the five Chapters is an independent unit, which may be accessed directly. The last Chapter is mainly addressed to the expert in the field, where the theory is well-known.

Even if the heart of the book starts with Chapter 5, the beginner should realize that not only Chapters 5, 6, and 7 are relevant and worth to study in this area, but the whole path throughout the various topics briefly discussed in previous chapters are necessary to appreciate the full implication of the theory of stochastic differential equation with jumps.

Michigan (USA), *Jose-Luis Menaldi*, April 2008

Contents

stochastic differential equation with jump component. stochastic integral on martingale. stochastic integral on Poisson measure. stochastic Taylor expansion. Stratonovich stochastic integral. strong approximation. strong convergence.Â Kuznetsov D. F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. Zh. Vychisl. Mat. i Mat. Phiz. 41: 6 (2001), 922-937. (In Russ.) Stochastic Differential Equations with Applications R. Situ. Forthcoming: The Fast Solution of Bounda~yIntegral Equations S. Rjasanow and 0. Steinbach. Theory of stochastic differential equations with jumps and applications mathematical and analytical techniques with applications to engineering. Rong situ. a-. Springer. Library of Congress Cataloging-in-Publication Data. Theory of Stochastic Differential Equations with Jumps and Applications: Mathematical and Analytical Techniques with Applications to Engineering By Rong Situ. ISBN-10: 0-387-25083-2 e-ISBN 10: 0-387-25175-8. Printed on acid-fr Learn more about bms with jumps.Â Tags. bms with jumps. Discover what MATLABÂ® can do for your career. Opportunities for recent engineering grads. Apply Today. An Error Occurred. Unable to complete the action because of changes made to the page. Reload the page to see its updated state.