A country of which nothing is known but the name
Grothendieck and "motives"

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To the memory of Monique Cartier (1932-2007)

Introduction

It is superfluous to introduce Alexander Grothendieck to mathematicians: he is recognized as one of the greatest scientists of the 20th century. For other audiences, however, it is important to explain that Grothendieck is much more than his rather sulphurous reputation, that of a man in a state of rupture, committing what one could call the suicide of his work, or at any rate consciously destroying the scientific school that he had created. What I want to discuss here are the interactions between his scientific work and his extraordinary personality. Grothendieck's story is not absolutely unique in the history of science; one may think of Ludwig Boltzmann for example. But there are essential differences: Boltzmann's work was rejected by the scientific community of his time and remained unrecognized until after his death, whereas Grothendieck's scientific work was immediately and enthusiastically accepted in spite of its innovative nature, and developed and continued by top-notch collaborators. The path traveled by Grothendieck appears different to me: a childhood devastated by the effect of Nazi crimes, an absent father who soon perished in the torments of the time, a mother who held her son in thrall and permanently affected his relationship with other women; all of this compensated for by an unlimited investment in mathematical abstraction, until psychosis could no longer be held off and came to drown him in the anguish of death – his own and the world's.

The case of Georg Cantor is an intermediate one, which has been beautifully analysed by Nathalie Charraud. After encountering violent opposition to his ideas, the support of great mathematicians such as Dedekind and Hilbert allowed him to reach an apotheosis at the International Congress of Mathematicians[1] in 1900 in Paris. The French school of analysis, from Poincaré to Borel, Baire and Lebesgue, was converted with enthusiasm to Cantor's ideas. Cantor's ultimate mental shipwreck may perhaps be attributed to the "Nobel syndrome", by which term I mean a type of depression which has been observed to occur in certain Nobel prize winners. Incapable of confronting their own individuality and the life that remains before them – especially when the prize has been attributed at a young age – with the world-renowned public figure they have now become, they fear that they have already given the best of themselves and will never again be able to reach the same height. There is an echo of self mockery in this feeling.

The typology of Grothendieck is incredibly complex. Like Gauss, Riemann, and so many other mathematicians, his major obsession was with the idea of space. But

[1] This is the official name of the world congress in mathematics that takes place every four years. Note the shift from mathematics to mathematicians in the title.

Grothendieck's originality was to deepen the idea of a geometric point[2]. As futile as such research might appear, it is nevertheless of considerable metaphysical importance, and the philosophical problems related to it are far from entirely solved. But what kind of intimate concerns, what secret fears are indicated by this obsession with the point? The ultimate form of this research, that of which Grothendieck was proudest, was that concerning the concept of a "motive", considered as a beam of light illuminating all the incarnations of a given object in its various guises. But this is also the point at which his work became unfinished: a dream rather than an actual mathematical creation, contrarily to everything else I will describe below in his mathematical work.

Thus, his work eventualy opened onto an abyss. But Grothendieck's other originality is that of fully accepting this. Most scientists are careful to efface their footprints on the sand and to silence their fantasies and dreams, in order to construct their own inner statue, in the words of François Jacob. André Weil was typical in this: he left behind a perfectly finished product in the classical style, in two movements: his *Scientific Works*, recently graced by a compelling *Commentary* written by himself, and a fascinating but carefully filtered autobiography, *Memories of an apprenticeship*, in which the effects of privacy and self-censorship are veiled by the appearance of a smooth and carefree tale.

Grothendieck played at a different game, nearer to Rousseau's *Confessions*. From the depths of his self-imposed retreat, of now nearly ten years – which it would be indecent to attempt to force – he sent us a vast introspective work[3]: *Récoltes et Semailles*. I will make use of this confession to try to clarify some of the main features of his work. But let us not fool ourselves: Grothendieck reveals himself in all his nakedness, exactly as he appears to himself, but there are clear signs of well-developed paranoia, and only a subtle analysis could reveal all the partly unconscious blockages and silences. The existence of *Récoltes et Semailles* aroused a somewhat unhealthy curiosity in the eyes of a certain public, akin to the sectarian devotion to a guru, an imaginary White Prince. For myself, I will stick to an analysis of the work and of the biography of the author, remaining as rational and honest as possible, before letting *Récoltes et Semailles* illuminate this exceptional body of work from within.

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Birth of the mathematical work

To present Grothendieck's scientific work in a few pages to a non-specialist audience

[2] On the occasion of the 40th anniversary of the IHES, a "Festschrift" was published as a special issue of the *Publications Mathématiques* which was not widely circulated. My contribution, entitled *La folle journée*, was an analysis of the notion of a geometric point, where Grothendieck's ideas are largely present. An English translation appeared in the *Bulletin of the AMS* (October 2001).

[3] Grothendieck was a very close friend, and we also collaborated scientifically, but I have not seen him in more than ten years. He sent me only a part of *Récoltes et Semailles*, the part that he judged I would be able to comprehend. For the missing part, I borrowed the copy belonging to the library of the IHES.

is something of a dare. To do it, I will make use of the analysis given by Jean Dieudonné – for years Grothedieck's closest associate – in his introduction to the "Festschrift" produced on the occasion of Grothendieck's 60th birthday[4].

The inheritance of Cantor's Set Theory allowed the 20th century to create the domain of "Functional Analysis". This comes about as an extension of the classical Differential and Integral Calculus (created by Leibniz and Newton), in which one considers not merely a particular function (for example the exponential function or a trigonometric function), but the operations and transformations which can be performed on all functions of a certain type. The creation of a "new" theory of integration, by Émile Borel and above all Henri Lebesgue, at the beginning of the 20th century, followed by the invention of normed spaces by Maurice Fréchet, Norbert Wiener and especially Stefan Banach, yielded new tools for construction and proof in mathematics. The theory is seductive by its generality, its simplicity and its harmony, and it is capable of resolving difficult problems with elegance. The price to pay is that it usually makes use of non-constructive methods (the Hahn-Banach theorem, Baire's theorem and its consequences), which enable one to prove the existence of a mathematical object, but without giving an effective construction. It is not surprising that a beginner, infatuated with generality, reacted with enthusiasm at what he learned about this theory in Montpellier, during the course of his undergraduate studies under somewhat old-fashioned professors. In 1946, Lebesgue's theory of integration was nearly 50 years old, but it was still hardly taught in France, where it was considered as a high precision tool, reserved for the use of especially able artisans [5].

Upon his arrival in the mathematical world of Paris, in 1948 at the age of 20, he had already written a long manuscript in which he reconstructed a very general version of the Lebesgue integral. Once he was received into a favorable milieu, in Nancy, where Jean Dieudonné, Jean Delsarte, Roger Godement and Laurent Schwartz (all active members of Bourbaki) were attempting to go beyond Banach's work, he revolutionized the subject, and even, in a certain sense, killed it. In his thesis, defended in 1953 and published in 1955, he created from scratch a theory of tensor products for Banach spaces and their generalizations, and invented the notion of "nuclear spaces". This notion, created in order to explain an important theorem of Laurent Schwartz on functional operators (the "kernel theorem"), was subsequently used by the Russian school around Gelfand, and became one of the keys of the application of techniques from probability theory to problems from Mathematical Physics (statistical mechanics, "constructive" quantum field theory). Grothendieck left this subject, after a deep and dense article on metric inequalities, which fed the research of an entire school (G. Pisier and his collaborators) for 40 years. But, in rather characteristic fashion, he never paid attention to the descendence of his ideas, and showed nothing but indifference and even hostility towards theoretical physics, a subject guilty of the destruction of Hiroshima!

Starting in 1955, at the age of 27, he began a second mathematical career. It was the

[4] "The Grothendieck Festschrift", 3 volumes edited by P. Cartier, L. Illusie, N.M. Katz, G. Laumon, Y. Manin and K.A. Ribet, Birkhäuser, Boston-Basel-Berlin, 1990.

[5] In the same period, Quantum Mechanics, the other pillar of twentieth century science, whose mathematical basis makes extensive use of functional analysis, was banished from French teaching for very similar reasons.

golden age of French mathematics, where, in the orbit of Bourbaki and impelled above all by Henri Cartan, Laurent Schwartz and Jean-Pierre Serre, mathematicians attacked the most difficult problems of geometry, group theory and topology. New tools appeared: sheaf theory and homological algebra (invented by Jean Leray on the one hand, Henri Cartan and Samuel Eilenberg on the other), which were admirable for their generality and flexibility. The apples of the garden of the Hesperides were the famous conjectures[6] stated by André

[6] Here is a simple presentation of the problem. Consider a prime number p and an equation of the form $y^2 = x^3 - ax - b$, where a and b are integers modulo p. We want to count the number N_p of solutions of this equation, where x and y are also integers modulo p. According to Hasse (1934), we have the inequality $|N_p - p| \leq 2\sqrt{p}$; this inequality has recently found applications in coding theory. In a result announced in 1940 and completely proven in 1948, André Weil considered the case of a more general equation, of the form $f(x, y) = 0$, where f is a polynomial with integral coefficients modulo p. Here the inequality takes the form $|N_p - p| \leq 2g\sqrt{p}$, where the new element is the integer g, the *genus* (which is equal to 1 in the case above). The genus is an algebraic invariant of the equation $f = 0$, whose significance was discovered by Riemann: the initial equation can also be written as a congruence $F(x, y) \equiv 0 \bmod p$, where the polynomial F has integer coefficients. Now we consider the set of solutions to the equality $F(x, y) = 0$ where x and y are *complex numbers*; these solutions form a "Riemann surface" which is obtained by adding g handles to a sphere.

The final inequality was proven by Weil and Lang in 1954: if we consider a system of m equations $f_1 = \cdots = f_m = 0$ in n variables $x_1, \ldots, x_n$, the number N_p of solutions satisfies an inequality $|N_p - p^d| \leq Cp^{d-1/2}$, where the integer d is the *algebraic dimension*, usually given by $d = n - m$. The constant C is more difficult to describe explicitly. But in the case above, we have $n - 2$, $m = 1$, $d = 1$ and $C = 2g$.

The challenge proposed by Weil in 1949 was to give an *exact formula*, not just an inequality. To do this, one has to count in N_p also the points at infinity (in the sense of projective geometry) of the variety V defined by the equations $f_1 = \cdots = f_m = 0$, giving a new number $\overline{N}_p$ of solutions. By a generalisation of the construction given above, in which the congruences modulo p were replaced by equalities of complex numbers, one can associate to V a space S of dimension $2d$, locally parametrized by d complex numbers (recall that for Riemann surfaces, we have $d = 1$, so $2d = 2$). The space S has geometric invariants called *Betti numbers*, denoted by $b_0, b_1, \ldots, b_{2d}$. Weil conjectured that

$$\overline{N}_p = S_0 - S_1 + S_2 - \cdots - S_{2d-1} + S_{2d}$$

$$S_i = a_{1,i} + \cdots + a_{b_i,i} \quad \text{with} \quad |a_{j,i}| = p^{i/2}, \quad \text{for} \quad i = 0, 1, \ldots, 2d.$$

In particular, we have $b_0 = b_{2d} = 1$, and $S_0 = 1$, $S_{2d} = p^d$. In the case of dimension $d = 1$, we have $b_0 = 1$, $b_1 = 2g$, $b_2 = 1$ and $\overline{N}_p = 1-(a_1+\cdots+a_{2g})+p$ with $|a_i| = \sqrt{p}$, from which we immediately deduce that $|\overline{N}_p - 1 - p| \leq 2g\sqrt{p}$ (but here $\overline{N}_p = 1 + N_p$ in the standard case). Weil then gave a complete treatment of a certain number of classical examples, in accordance with this conjecture, and Chevalley applied these counting methods to the theory of finite groups.

Weil in 1954: these conjectures appeared as a combinatorial problem (counting the number of solutions of equations with variables in a Galois field) of a discouraging generality (even though several meaning special cases were already known). The fascinating aspect of these conjectures is that they assume a sort of *fusion of two opposite poles*: "discrete" and "continuous", or "finite" and "infinite". Methods invented in topology to keep track of invariants under the continuous deformation of geometric objects, must be employed to enumerate a finite number of configurations. Like Moses, André Weil caught sight of the Promised Land, but unlike Moses, he was unable to cross the Red Sea on dry land, nor did he have an adequate vessel. For his own work, he had already reconstructed "algebraic" geometry on a purely algebraic basis, in which the notion of a "field" is predominant. To create the required "arithmetic" geometry[7], it is necessary to replace the algebraic notion of a field by that of a commutative ring, and above all to invent an adaptation of homological algebra able to tame the problems of arithmetic geometry. André Weil himself was not ignorant of these techniques nor of these problems, and his contributions are numerous and important (adeles, the so-called Tamagawa number, class field theory, deformation of discrete subgroups of symmetries). But André Weil was suspicious of "big machinery" and never learned to feel familiar with sheaves, homological algebra or categories, contrarily to Grothendieck, who embraced them wholeheartedly.

Grothendieck's first foray into this new domain came as quite a thunderclap. The article is known as "Tohoku", as it appeared in the *Tohoku Mathematical Journal* in 1957, under the modest title "Sur quelques points d'algèbre homologique". Homological algebra, conceived as a general tool reaching beyond all special cases, was invented by Cartan and Eilenberg (their book "Homological Algebra" appeared in 1956). This book is a very precise exposition, but limited to the theory of modules over rings and the associated functors "Ext" and "Tor". It was already a vast synthesis of known methods and results, but sheaves do not enter into this picture. Sheaves, in Leray's work, were created together with their homology, but the homology theory is constructed in an *ad hoc* manner imitating the geometric methods of Elie Cartan (the father of Henri). In the autumn of 1950, Eilenberg, who was spending a year in Paris, undertook with Cartan to give an axiomatic characterization of sheaf homology; yet the construction itself preserves its initial *ad hoc* character. When Serre introduced sheaves into algebraic geometry, in 1953, the seemingly pathological nature of the "Zariski topology" forced him into some very indirect constructions. Grothendieck's flash of genius consisted in solving the problem from above, as he would do again and again in the years to come. By analysing the reasons for the success of homological algebra for modules, he unearthed the notion of an abelian category (invented simultaneously by D. Buchsbaum), and above all the condition he

[7] The German geometer Erich Kähler published an article in 1958 (in Italian) entitled "Geometrica Arithmetica" (Annali di Matematica, t. XLV, 368 pages), and the name was an immediate success. The domain is also called "Diophantian analysis", after the Greek mathematician Diophantus. It is the theory of polynomial equations such as $x^3+y^3+z^3 = t^3$, which possess an infinite number of solutions in real or complex numbers, but for which one restricts oneself to seeking the integer (or rational) solutions. The very existence of such solutions is then in question, and the properties of divisibility and prime numbers play a large role, giving an "arithmetic" character to the subject.

labels as AB5*. This condition guarantees the existence of "enough injective objects". The sheaves satisfying this condition AB5*, and along with it, the method of injective resolutions which is fundamental for modules, extends to sheaves *without need for any artifice*. Not only does it give a sound basis for the construction of sheaf homology, but it provides an absolutely parallel development for modules and sheaves, bringing the Ext and Tor functors over to sheaves. Everything has now become entirely *natural*.

After this first "initiation" (1955-58), Grothendieck stated in 1958 his research program: to create arithmetic geometry via a (new) reformulation of algebraic geometry, seeking maximal generality, appropriating the new tools created for the use of topology and already tested by Cartan, Serre and Eilenberg. He dared attack the synthesis that none of the actors of the time (Serre, Chevalley, Nagata, Lang, myself) had dared, throwing himself into it with his own characteristic energy and enthusiasm. The time was ripe; world science was living its most intense phase of development during the 1960's, and the disenchantment of the years following the 1968 social movement had not yet begun. Grothendieck's undertaking throve thanks to unexpected synergies: the immense capacity for synthesis and for work of Dieudonné, promoted to the rank of scribe, the rigorous, rationalist and well-informed spirit of Serre, the practical know-how in geometry and algebra of Zariski's students, the juvenile freshness of the great disciple Pierre Deligne, all acted as counterweights to the adventurous, visionary and wildly ambitious spirit of Grothendieck. The new Institut des Hautes Études Scientifiques (IHES), created for him and around him, set in motion a constellation of young international talent. Organised around the key notion of a "scheme"[8], Grothendieck's theory ended up annexing every part of geometry, even the newest parts such as the study of "algebraic groups". Using a gigantic machine: Grothendieck topologies (etale, crystalline,...), descent, derived categories, the six operations, characteristic classes, monodromy and so on, Grothendieck arrived halfway down the path he had set himself, whose final goal was the proof of the Weil conjectures. In 1974, Deligne put the final touch on the proof, but in the meantime, Grothendieck had dropped everything since 1970, after 12 years of a undisputed scientific reign over the IHES.

What were the reasons for this total abandonment in the middle of everything? Put bluntly, his psychosis caught up with him, but at the time, it was stimulated by more direct reasons: the despair of being surpassed by his favorite disciple Deligne, the "Nobel syndrome", the revelation by the "1968 revolution" of the contradiction between the free spirit he believed himself to be and the university "mandarin" he appeared in the eyes of others, a feeling of failure faced with some of his aborted mathematical efforts (the Hodge conjecture, the standard conjectures), weariness and exhaustion after 20 years of total devotion, day and night, to the service of his mathematical muse? A mixture of all of those.

It remains to make some observations on Grothendieck's "posthumous" work. After his break with the mathematical world, which essentially occurred at the ICM in Nice (September 1970), and two further years of *Wanderung*, he became an ordinary professor

[8] This word results from a typical epistemological shift from one thing to another: for Chevalley, who invented the name in 1955, it indicated the "scheme" or "skeleton" of an algebraic variety, which itself remained the central object. For Grothendieck, the "scheme" is the focal point, the source of all the projections and all the incarnations.

at the very same average-level university (Montpellier) where he had studied as an undergraduate. He had a few more students, none of which attained the level of his team at the IHES, and whose fortunes were varied. Until his official retirement, in 1988 at the age of 60, he continued to work at mathematics in occasional spurts, leaving a "posthumous" body of work not without importance. There are three main texts:

– *Pursuing Stacks* (written in 1983) is a 600-page reflection on higher categories. Combinatorics, geometry and homological algebra come together in a grandiose project. After more than 15 years of the combined efforts of many, three (probably nearly equivalent) definitions have been proposed for multidimensional categories (in the widest sense[9], using a cascade of composition laws. For general categories (called "lax"), the point is the following: when one wants to formulate an identity at a certain level, say $A = B$, one has to create a new object on the level just above, which realizes the transformation from A to B. It is a kind of dynamic theory of relations. In spirit, it is analogous to the Whitehead-Russell type of theory, but with a geometric aspect; in fact, Grothendieck conceives of his "stacks" as generalisations of homotopy theory (which studies deformations in geometry). The fusion of logic and geometry whose beginnings are visible in the theory of stacks and toposes, is one of the most promising directions indicated by Grothendieck. Their importance is not just for "pure" mathematics, since a good theory of "assemblages" would have many potential applications in theoretical computer science, statistical physics, etc.

– The *Esquisse d'un Programme* was a text written in 1984 for inclusion in the application for a position with the CNRS. In it, Grothendieck sketches (the word is exact) the construction of a tower (or a game of Lego) describing deformations of algebraic curves.

– *The long march through Galois theory*, written before the previous one (in 1981), gives partial indications about some of the constructions suggested in the *Esquisse*.

These texts have circulated by being passed from hand to hand, with the exception of the *Esquisse* which was finally published, thanks to the insistence of a group of "devotees". Curiously, the true heirs of Grothendieck's work are essentially members of a Russian mathematical school (Manin, Drinfeld, Goncharov, Kontsevitch, to cite just a few), who have had little if any direct contact with Grothendieck, but who inherited and made use of methods from mathematical physics – a domain which he loathed and of which he was totally ignorant.

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Biographical elements

The first thing is to describe something of Grothendieck's family origins, in order to place him in a proper perspective. There were three central characters: the *father*, the *mother* and the *son*, each remarkable in his own way, and a ghost – an older half-sister, on the mother's side, who died recently in the United States, and whom he did not know

[9] It is not difficult to define a *strict* multidimensional category

very well[10].

According to my information, the father's name was Shapiro – which indicates a Jewish origin. He was apparently born in Belyje-Berega, which is today situated in Russia, near the border with Belorussia and the Ukraine, now independent countries. At the time, it was a Jewish town located in the Ukraine and inhabited by very pious Hassidic Jews. Breaking away from this background, Shapiro frequented the revolutionary Jewish circles in Russia, and took part at the very young age of 17 in the aborted revolution of 1905 against the tzars. He paid for this participation with more than 10 years in prison, and was only freed at the 1917 revolution. This was the beginning of a long period of revolutionary wandering, and the first of a long series of imprisonments. His son told me one day, with pride and exaltation, that his father had been a political prisoner under 17 different regimes. I answered that he should have been included in the *Who's Who* of the Revolution, and he didn't deny it! But a sign of the Bolshevik taboos which still exist is that in fact, most of the histories of socialism – including the ones written by Trotskyists like Pierre Broué – give virtually no information about Shapiro or his companions. There is still quite a bit of historical research to be done there.

According to what I know, in 1917, he belonged to the left-wing S.R. (Revolutionary Socialists), one of the factions which was struggling for power in Saint-Petersburg. We know that in the end, Lenin crushed all the factions except for the Bolsheviks, not to mention their own internal purges. One of the best descriptions of these events, although obviously partially romanticized, is the famous book by John Reed: *Ten Days that Shook the World*. Grothendieck always told me that one of the people in the book was his father. After Lenin's purges, Shapiro was to be found everywhere that an extreme left-wing revolution broke out in Europe in the 1920's – and there were many! Naturally, he was with Bela Kun in Budapest, with Rosa Luxemburg in Berlin, with the Soviets in Munich. When Nazism was on the rise in germany, he struggled with the S.A.P. (Left-wing Socialist Party) against the Nazis, and he was compelled to leave Germany when Hitler came to power. Then, naturally, he was to be found in the Spanish Civil War, in the International Brigades (with the P.O.U.M – worker's Marxist unification party) like Simone Weil, in a surprising parallel. After Franco's victory in Spain, he joined his wife Hanka and their son Alexander, refugees in France.

The end of his story is a manifestation of the shame of our country. When he returned to France, he was a broken man, according to his son. He drifted without energy for a while, and then, like so many other antifascist refugees, emigrants from Germany or Spain, he was interned, early in 1939, in the Camp du Vernet. This was not, of course, an extermination camp, although many of the prisoners died of malnutrition or lack of medical care (for example in Gurs). But what exactly are the differences between a refugee camp , an internment camp and a concentration camp[11]? In any case, without ever recovering his freedom, he was handed over to the Nazis by the Vichy authorities, and finally perished in

[10] Is it merely a coincidence that there was also such a ghost in Einstein's life: a girl, born before his first marriage, whose trace has been entirely lost as neither of the two parents wanted to find their child?

[11] My colleague Szpiro has confirmed this point; his father was interned at Vernet for analogous reasons. Sixty years later, the testimonies begin to emerge!

Auschwitz. The last concrete sign of his life that exists is a rather hallucinatory portrait in oils, painted by another prisoner in Vernet, which his son preserved like a talisman – the similarity is so striking that it could almost be a portrait of the son.

Hanka Grothendieck – that was the name of Alexander's mother – came from Northern Germany. In the 1920's, she militated in various left-wing groups, and tried to be a writer. She had a daughter, mentioned above, and then met Shapiro, and Alexander was born in Berlin[12] in March 1928. She emigrated to France when Hitler came to power, and managed to scrape a survival in the circles of German emigrants, which Simone Weil frequented around that period. In September 1939, when war was declared, the situation of these refugees, already very difficult, became worse, as they were henceforth considered "enemy citizens". In any case, Hanka and her son were interned in Mende in 1939, and their situation was not eased until after the catastrophe of June 1940.

Alexander – he cared about using this spelling rather than the French "Alexandre" – was left behind by his parents when they left Germany. He remained hidden on a farm in Northern Germany until about 1938, when he was 10 years old, raised by a teacher in the style of Freinet, who believed in a "return to Nature". This "natural" ideology (inherited from Romanticism) was shared by the most diverse political groups in Germany, from the Nazis to the Socialists, and anticipated the concerns of ecological groups fifty years later. But he preferred to talk about the period of his life that he spent in le Chambon-sur-Lignon, from 1942 to 1944. The true nature of the resistance in the Cévenol region is much better understood nowadays. Le Chambon-sur-Lignon, an agreeable village and vacation site frequented mainly by Protestants, has a private high school called Collège Cévenol, which until 1939 nothing more than a prep school for wealthy Protestant youth. During the war, however, the decisive hand of Pastor Trocmé transformed the Collège Cévenol into the center of a movement of spiritual resistance to Nazism, consonant with the historical military resistance rooted deep within the Huguenot tradition, which performed splendid feats of rescue for the Jewish children who came under its care. Grothendieck was a boarder at the Foyer Suisse and a student at the Collège, and made an impression so strong that even at the end of the 1950's I was able to obtain some personal memories from people who recalled him.

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His childhood ended there. Thanks to the Collège Cévenol, he obtained his baccalaureate and became a student in Montpellier in 1945. Then began the period of his *scientific training*. With the help of *Récoltes et Semailles*, I will now examine from the inside the gestation of the mathematical work that I described from the outside in the paragraphs above.

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[12] The *Götterdämmerung* of Berlin in 1945 saw the destruction of all public records. Because of this, Grothendieck had continual administrative problems. Until the beginning of the 1980's, he had to travel with a "Nansen passport" from the United Nations, documents which were parsimoniously offered to stateless people. After 1980, convinced that he could no longer be called up to serve in the French army, he consented to apply for French citizenship.

It was in Montpellier, during his undergraduate days, that he underwent his first real mathematical experience. He was very unsatisfied with the teaching he was receiving. He had been told how to compute the volume of a sphere or a pyramid, but no one had explained the definition of volume. It is an unmistakeable sign of a mathematical spirit to want to replace the "how" with a "why". A professor of Grothendieck assured him that a certain Lebesgue had resolved the last outstanding problems in mathematics, but that his work would be too difficult to teach. Alone, with almost no hints, Grothendieck rediscovered a very general version of the Lebesgue integral. The genesis of this first mathematical piece of work, accomplished in total isolation, is beautifully described in *Récoltes et Semailles*: he discovered that he was a mathematician without knowing that there was such a thing as a mathematician. Of course, he was surrounded by mathematics students and professors who taught mathematics properly enough, but who could not be taken for mathematicians: in all simplicity, he thought he was the only one in the world.[13]

Grothendieck's "public period" (as we speak of the "public period" of that other rabbi, Jesus) began on his arrival in Paris in 1948, with a bachelor's degree in his pocket. His professor from Montpellier, who had much earlier completed a master's degree with Élie Cartan, had given him a letter of recommendation to his old teacher. He was unaware that Élie Cartan, three years before his death, was much diminished, and that his son Henri, a mathematician as famous as his father had been, was now the dominating figure on the Parisian – and French – mathematical scene.

But there was not too much chemistry between the eminent Protestant university professor and the young self-taught rebel. André Weil suggested sending Grothendieck to Nancy, where Jean Delsarte, one of the founding fathers of Bourbaki and a skilled organizer, had pushed the department of which he was Dean into becoming the first step in Bourbaki's march towards the conquest of the universities. Jean Dieudonné and Laurent Schwartz were able to discipline Grothendieck just enough to prevent him from running off in all directions, and to restrain his excessive attraction to extreme generality. They gave him problems which led him in the direction of his first work on the Lebesgue integral. It would be an understatement to say that the disciple surpassed his masters: he pulverized the domain of Functional Analysis via a *solitary work* during the course of which he had no companions, and which subsequently found no continuers.

It was in Nancy, also, that he became an adult in the popular sense of the word. From a relationship with his landlady, a son, Serge, was born. Serge had several older half brothers and sisters, and later on, when Grothendieck conceived the desire to take care of Serge himself, he was quite ready to adopt the entire family. He flung himself into a lawsuit to obtain paternal custody which was very unlikely to succeed – and which he sabotaged even further by insisting on taking advantage of the legal possibility to act as his own counsel. This was only the beginning of his chaotic family life: in all, he had five

[13] I experienced some of the same feelings during my provincial youth (in Sedan): I had a taste for mathematics, but I wasn't aware that they could actually constitute a profession. With an grandfather who had graduated from the engineering school Arts-et-Métiers and an uncle from the engineering school École Centrale, the ambition of my family was to see me enter the École Polytechnique! I would have been continuing the dynasty of engineers in the family, and that, it seemed, was the purpose of mathematics!

children from three mothers, and was as absent from their lives as his own father had been from his.

His mathematical work in Nancy made him famous, and he could have continued along the path he started on there. But he described himself very well as a builder of houses which it was not his vocation to inhabit. He embarked on the customary career of a researcher, recruited and promoted by the CNRS, then spending a few years abroad after his thesis. But when he returned from São Paulo, he had closed the chapter on Functional Analysis.

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That was the beginning of his *great period*, from 1958 to 1970, which coincides with the heyday of Bourbaki. What precipitated this period was the daring idea of Léon Motchane, who flung himself heart and soul into the adventure of creating the IHES (Institut des Hautes Études Scientifiques) in Bures-sur-Yvette. Léon Motchane, who had dreamed of being a mathematician, already had a successful career in business behind him, but he wanted to create something that would survive him. Dieudonné, who had left Nancy to spend some years in the United States, wanted to return to France. Motchane offered him the first chair in mathematics in the future institute, and Dieudonné accepted on condition that he hire Grothendieck as well. He himself was at a turning point in his career: he had reached the key age of 50 at which members of Bourbaki were required to quit the group, and he had already produced his most original piece of research, on "formal groups". Dieudonné, who was at heart a man of order and tradition[14], placed himself for the second time at the service of a revolutionary enterprise: after Bourbaki, the dual adventure of Motchane and Grothendieck.[15]

In an extraordinary organisation of division of labor, the young Grothendieck created one of the most prestigious mathematical seminars that has ever existed. He attracted all the talented students[16] and threw himself with passion into mathematical discovery, in sessions that lasted ten or twelve hours (!) He formulated a grandiose program destined to fuse arithmetic, algebraic geometry and topology. A builder of cathedrals according to his own allegory, he distributed the work amongst his teammates. Every day, he sent an interminable pile of illegible notes to the elder Dieudonné who, sitting at his worktable from 5 to 8 o'clock each morning, transformed the scribbles into an imposing collection of volumes signed by both Dieudonné and Grothendieck, which came out in the "Publications Mathématiques" of the IHES. Dieudonné had no personal ambitions and placed himself

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[14] We used to have epic political discussions, during which he attacked what he called my "communism", which was really nothing more than an adherence to "progressive Christian" tendencies: the nuance escaped him, but maybe he was right after all.

[15] Though with very different destinies, both Motchane and Grothendieck were heirs of the revolutionary Jews of Saint-Petersburg under the tzar. And in fact, both of Léon Motchane's sons became left-wing militants.

[16] In *Récoltes et Semailles*, Grothendieck counts his *twelve* disciples. The central character is Pierre Deligne, who combines in this tale the features of John, "the disciple whom Jesus loved", and Judas the betrayer. The weight of symbols!

entirely at the service of this work as selflessly as he had done for the work of Bourbaki. Dieudonné did not remain at the IHES for many years; upon the the creation of the University of Nice, he accepted the position of Dean of Sciences. But that did not stop his collaboration with Grothendieck. Only in 1970, when Dieudonné, no longer young, yet found the energy to organise the International Congress of Mathematicians in Nice, was the absolute rupture between the two mathematicians consummated[1]7

The legendary duo was, in fact, a trio. Jean-Pierre Serre, with his sharp sense of mathematics, his deep and broad mathematical culture, his quickness of thought and his technical prowess, was always there, protectively. He acted as an intermediary between Weil and Grothendieck when they no longer wished to communicate directly, and contributed greatly to the clarification of the above-mentioned Weil conjectures. At a time when the rate for suburban telephone calls was the same as the local rate, Serre and Grothendieck talked between Bures and Paris for hours each day. Serre was the perfect beater (I was going to say matchmaker), scaring the mathematical prey straight into Grothendieck's nets – and in nets as solid as those, the prey did not resist long.

Their success was immediate and smashing. As early as 1962, Serre was declaring that algebraic geometry was one and the same with scheme theory [18]. Publications on the subject, direct or indirect, grew into the thousands of pages; every newcomer to the domain needed to have read everything, and forty years later, a simple and concise yet complete exposition of the entire subject still does not exist. As Grothendieck described in his allegories, a certain know-how risks disappearing altogether, from lack of fresh blood. After Grothendieck's departure from mathematics, Deligne and Illusie did a masterly archival job in completing the publication of the "Séminaire de Géométrie Algébrique", but Grothendieck was not grateful. It is true that what remains of Grothendieck's school has become a closed circle; a certain generosity has been lost, a certain breeze has died away – but then, the same is true for Bourbaki.[19]

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But the Tarpeian Rock lies not far from the Capitol! Grothendieck's scientific fame reached its peak in 1966. At the International Congress of Mathematicians, he was awarded the crowning honor: the Fields Medal[20]. The Soviet authorities were not very eager to give him a visa (his father had become an "enemy of the people" after the 1917 revolution). This was the time of the Vietnam War, and many mathematicians were against the war; not only Grothendieck but also, for example, Steve Smale, another winner of the Fields Medal in 1966. In the context of the Cold War between the USSR and the USA, which was particularly virulent at that time, certain Soviets may have hoped to make use of

[1]7 At that point there was nothing but total incomprehension between the believer in science for the sake of science and the libertarian militant who wanted to use the Congress as a tribunal for his ideas.

[18] Grothendieck's creation!

[19] Common destiny of institutions and civilizations!

[20] Which is often compared to the Nobel Prize (which doesn't exist in mathematics), but is limited to 3 or 4 laureates every 4 years.

these mathematicians. But the press conference organized by Steve Smale in Moscow[21] (during which he not only denounced the Vietnam War but also compared it to the Soviet invasion of Hungary) must have shown them that mathematicians are not always easy to manipulate.

If I am allowed, as an absolute novice in the domain of psychoanalysis, to formulate a hypothesis, *it was in Moscow that the abyss opened for Grothendieck*, or rather, his fundamental wound reopened. This wound was that of the *absent father*, victim of Stalinists and Nazis, the Russian Jewish father recalled by the connection with a country in which anti-Semitism underwent a significant revival in the 1960's (if it had ever actually disappeared). Of course, there is also what I termed the "Nobel syndrome" above, and Grothendieck must certainly have said to himself that the Medal crowned an unfinished achievement, and suspected that he would never arrive at the end of his scientific ambitions.

Around this time, the great social rupture began in France, which – following the feverish atmosphere in Berkeley in 1965 – led to the famous events of May 1968[22]. Grothendieck had not become politically involved in the Algerian war[23], and perhaps he wished to make up for that, and felt the pressure of the revolutionary past of the father he so admired. In any case, the social rupture revealed to him his own inner contradictions. Certainly Grothendieck had known what it was to be "undesirable", to be interned in camps, and he always lived very modestly, even when he was one of the gods of international mathematics. He was always very attentive to homeless people and to those who are flung to the side of the road by the march of society; his house was always a sort of court of miracles (which was not always easy for his family life), and he had not forgotten the difficult years of his childhood.

Yet, in 1968, he – whose mental self-image was anchored in the identity of the outlaw, the anarchist – suddenly discovered that he was a revered pontiff of international science, invested with great authority over both ideas and people. During this period, in which all authority was challenged, even intellectual authority, he became aware of the coexistence of two personalities within himself, and that was the beginning of a time of wavering which lasted for four or five years. His temporary response was to found a tiny group, which put forth a newsletter called *Survivre* and later *Survivre et Vivre*. This movement resembled one of those ecolo-catastrophe-oriented sects which sprang up everywhere in the 1970's: the danger (quite real at the time) of a nuclear war worked together with obsessions about pollution and overpopulation. The integral pacifism inherited from his father expressed itself within "Survive", and he put all of his scientific fame to use in the furtherance of his ecological aims. He surely believed that social issues can be settled with the same kind of

[21] Of which Grothendieck would have very probably approved – however he elected not to set foot in Moscow, having his Medal formally collected by Léon Motchane in his stead.

[22] Now that the "events in Algeria" have been officially rebaptized "the Algerian War", which they were, perhaps we can look for an adequate name for the "events of 1968"?

[23] His insistence on not becoming a French citizen had enabled him to avoid being drafted during the Algerian war, but he paid a price. I only remember that he asked me once, in the early 60's, why I had not deserted. I myself took part in that cursed war, even though it was only for a short time.

proofs as mathematical ones, and in general he ended up actually irritating people even when they were aware of his importance as a mathematician, and perfectly receptive to the ideas he was expressing. I recall two quite painful incidents, one in Nice in 1970 and the other in Antwerp in 1973, during which his deliberately provocative attitude ruined the patient efforts of others who had been working in the same direction as he was, but with a more political vision.

This period of Grothendieck's life was followed by a few years of wandering: he resigned from the IHES in September 1970, on a rather minor pretext[24]; travels abroad, a temporary position at the Collège de France[25], and finally, his accepting a position as professor at the University of Montpellier, the university of his youth, for which he felt but moderate esteem.

From his years in Montpellier, one particular event stands out: that of his *trial*. As I already said, Grothendieck was very welcoming to the rejects of society. In the 1970's, the regions of Lozère and Larzac became a kind of Promised Land for numerous hippy groups, and seen from the outside, Grothendieck's house must have resembled a phalanstery with himself as the guru. Following some real or exaggerated incidents, the local police was becoming nervous, and one day they raided Grothendieck's house. The only "offence" they could pin on him was the presence of a Japanese Buddhist monk, a former mathematics student at the Tata Institute in Bombay and a most inoffensive person, but whose residence permit in France had expired three weeks earlier. This was the kind of problem that a university professor can usually settle quite easily with a few contacts in the right places, but Grothendieck's philosophy prevented him from adopting this approach. The unexpected result was a summons to the Magistrate's Court of Montpellier six months later, the Japanese monk having in the meantime disappeared to the antipodes. Was it a preliminary test of Pasqua's laws? Or did the local authorities think that Grothendieck was a suspicious hippie? What ought to have been a quickly expedited ten-minute procedure blew up into a major event. Grothendieck appeared at the Bourbaki Seminar in Paris in order to alert some of his colleagues to the situation; in particular Laurent Schwartz, Alain Lascoux and myself. We set in motion some activity: string-pulling in some intellectual groups, mobilization of a few networks, calling upon the League of Human Rights. On the day of the trial, the judge had received 200 letters in favor of the accused, and a specially chartered airplane disgorged a medley of supporters wearing Dean's robes (with

[24] The discovery of modest financial support given to the IHES, on the recommendation of Michel Debré, by the D.R.E.T. (an organization financing military research). The financial support of the IHES was quite opaque for a long time, but military funding never played anything more than a modest role. It isn't totally absurd, however, to imagine that there might have been a world plan for the drafting of scientists into a new world war (this time against the USSR), and the the IHES might have been part of that network. Only Léon Motchane could have answered the question.

[25] He was an "associated professor" (a position reserved for foreigners) there from 1970 to 1972. At the moment at which he could have received tenure, he explained clearly that he would use his chair as a vehicle for his ecological ideas. This resulted in a curious three-way competition between Grothendieck, Tits and myself, very unusual for the Collège de France, which ended by Tits being nominated to a chair in Group Theory.

Dieudonné at their head), or the robes of prominent lawyers. Grothendieck, who was appearing in court for the second time, had once again decided to act as his own lawyer. He gave a magnificent speech for the defence, which I still have somewhere. Naturally quoting Socrates, he concluded with the following exhortation: "I am being prosecuted in the name of a law passed in 1942 against foreigners. I was interned during the war in the name of this law, and my father died in Auschwitz because of it. I am not afraid of prison. If you apply the law, I may go to prison for two years. I am legally guilty and therefore will accept the punishment. But on a deeper level, I plead innocent. It is up to the judge to choose: the letter of the law and prison, or universal values and freedom. This was followed by a "setting the argument in legal form" by the lawyer Henri Leclerc, who later became the president of the League of Human Rights. This was the result of a laboriously negotiated compromise with Grothendieck, who would have preferred condemnation to compromise. Alas, as Grothendieck had predicted, the judge yielded to pressure, and compromised by giving him a suspended sentence of six months' imprisonment. The sentence was confirmed on appeal, but by then the public emotion had died away.

As I already said, he retired in 1988, and has lived since then in self-imposed exile. At first he lived near the Fontaine de Vaucluse, in the middle of a little vineyard that he cultivated, and near to his daughter Johanna and his grandchildren. But later he broke off every family relation. He didn't seem to mind that the place where he lived was located so near to the infamous Camp du Vernet which played a sad role in his childhood. He lived for years without any contact with the outside world and only a few people even knew where he was. He chose to live alone, considered by his neighbors as a "retired mathematics professor who's a bit mad". He expressed his spirituality through a series of experiments, in the Buddhist tradition and others; perhaps his orthodox Jewish ancestry played some role in his adherence to strict dietary rules. He was at one time an extreme vegetarian, to the detriment of his health. The parallel with the destiny of Simone Weil is a double one: the desire to be at the level of the poorest of the poor, and a kind of mental anorexy. It is conceivable that his end, like hers, could result from a complete refusal to eat.

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Autopsy of his work

Grothendieck's mathematical work in algebraic geometry totals more than 10,000 pages, published in two series. The first one, entitled *Elements of Algebraic Geometry* (EGA) with reference to Euclid's *Elements* was written entirely by Dieudonné, and has remained incomplete since only 4 parts have been completely written, out of an initially projected 13. The second series is called *Seminar of Algebraic Geometry* (SGA) and consists in 7 volumes. The composition of SGA is less regular. At the start, there were the Seminars in the Bois-Marie (from the name of the domain where the IHES eventually found its home), which he led from 1960 to 1969. The first two volumes were written by Grothendieck or under his control, and he directed their publication: the third seminar was essentially written by Pierre Gabriel and Michel Demazure (whose thesis was part of the work). Afterwards, things became more complicated. When Grothendieck abandoned mathematics in 1970, he left an incomplete worksite behind, and it was, in fact, a worksite in a pitiful state. There were manuscripts (literally) by Grothendieck which were difficult to decipher, mimeographed lectures from the seminar, and notes ready for publication. It would have been necessary to make a synthesis, plug the (sizeable) holes, and furthermore undertake an enormous writing job; all rather ungrateful tasks which would not bring any particular glory to their author. All was done, in the end, with faithfulness and filial piety, by Luc Illusie and Pierre Deligne. The central piece, in view of the Weil conjectures, is SGA 4, devoted to the most innovative ideas (toposes in particular; I will talk more about them below). In fact, when Deligne announced his complete proof of the Weil conjectures in 1974, experts considered that the foundations were insufficient, and (at the same time as the missing link of the Grothendieck seminar, SGA 5), he published an additional volume, essentially due to himself, under the curious name of SGA 4 1/2. Grothendieck took this new publication very badly, and took advantage of it to denigrate the entire enterprise: naturally, since it was not what he himself had in mind, his plans had been truncated, he had been betrayed... He describes this with a very strong image: the team of builders who, with their Master dead, disperse, each one carrying away his own sketches and tools. It is a beautiful image, but it has one problem: here, the Master abandoned his team, by deliberately committing suicide. I will now undertake the autopsy of this "assassinated" work.

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As Grothendieck possessed a taste for symbolism, he recognized exactly *twelve* disciples: to arrive at this number, he cheated a little, because there is no real mathematical definition of a disciple, and he forgets the "posthumous" disciple (Z. Mebkhout), whom he welcomed and later rejected, during the course of a rather inglorious polemic. In *Récoltes et Semailles*, he groups his work into *twelve* themes: I will not list them here, but I will comment on some of them.

The first theme he mentions is that of his thesis: *Functional Analysis*. He says himself that retrospectively, it seems to him rather like a school exercise, an intellectual warm-up. Certainly the perspective that Grothendieck gave to Functional Analysis is no longer modern; the major problems from within the theory have been solved, mostly by Grothendieck, and the subject has become one that serves others, its methods used to nourish the subjects of Fourier analysis (or its more recent form of wavelets) and partial differential equations. Grothendieck was pulled along by the current of "qualitative" topology of the time (which was very suited to his temperament), but today "quantitative" methods are more appreciated[26].

[26] His last text on the subject, which appeared in the Bulletin of the Mathematical Society of São Paulo in 1956, appears at first as a study of functors between Banach spaces (a premonition of his later investment in the theory of categories), even though the term "functor" does not appear. The central result is formulated as the equivalence of two of these functors. In a Bourbaki seminar, I reformulated his result as an inequality concerning matrices (with the "Grothendieck constant"), a "quantitative version" which was the starting point of later work (by G. Pisier). But Grothendieck did not accept my reformulation and considered himself betrayed.

But of course, all the other themes concern Grothendieck's grand enterprise: *algebraic geometry*. One of the sources of mathematical development consists of the great problems, the great enigmas whose relatively simple formulation doesn't give any place to latch on and get started. What was improperly known as Fermat's Last Theorem was a conjecture of Biblical simplicity, expressed in symbols as: "the relation $a^n + b^n = c^n$ is impossible if a, b, c, n are non-zero integers unless $n = 2$". It was proven recently (by A. Wiles and R. Taylor), via the construction of a large and complex edifice, largely based on methods due to Weil and Grothendieck. Now, the most prestigious and most perplexing open problem is the *Riemann hypothesis*. These two problems, Fermat and Riemann, are in some sense rather futile: Fermat's problem concerned a very particular equation, and the Riemann hypothesis can be interpreted in terms of very subtle regularities in the apparently random distribution of prime numbers. In itself, a counter-example to the Riemann hypothesis, given the present state of our knowledge, would have very small "practical" consequences and would certainly not be a catastrophe.

What is important to us here is a certain perception of the problems. Faced with the impossibility of proving the Riemann hypothesis, we have fled ahead. Following Artin and Schmidt, Hasse in 1930 formulated and solved a problem analogous to the Riemann hypothesis by translating it into the form of an inequality (see note [6]). The next step occupied Weil from 1940 to 1948. In all of these cases, by analogy with the Riemann zeta function, related to prime numbers[27], one associates zeta functions to the most varied geometric and arithmetic objects, and then off one goes on the way to proving the property analogous to Riemann – it has been done frequently with great success! All of these zeta functions have contributed greatly to structuring the field of arithmetic, and Weil was guided by these ideas when he formulated his conjectures in 1949. Weil was a classical mind, attached to clarity and precision, and his conjectures have these characteristics. But for Grothendieck, the Weil conjectures are interesting rather as a test of his basic vision than in and for themselves. Grothendieck distinguished between mathematician-builders and mathematician-explorers, but saw himself as being both at the same time (André Weil was certainly less of a builder than Grothendieck, and he detested "big machinery" even if he had to construct it on occasion).

Grothendieck's favorite method is not unlike Joshua's method for conquering Jericho. The thing was to patiently encircle the solid walls without actually doing anything: at a certain point, the walls fall flat without a fight. This was also the method used by the Romans when they conquered the natural desert fortress Massada, the last stronghold of the Jewish revolt, after spending months patiently building a ramp. Grothendieck was convinced that if one has a sufficiently unifying vision of mathematics, if one can sufficiently penetrate the essence of mathematics and the strategies of its concepts, then particular problems are nothing but a test; they do not need to be solved for their own sake.

[27] Let us recall one of its definitions:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

where the product runs over all prime numbers $p = 2, 3, 5, 7, 11, ...$

This strategy worked very well for Grothendieck, even if his dreams tended to make him go too far at times, and he needed the correcting influence of Dieudonné and Serre. But I already explained that Grothendieck only went three-quarters of the way, leaving the final conclusion to Deligne. Deligne's method was totally perpendicular to Grothendieck's: he knew every trick of his master's trade by heart, every concept, every variant. His proof, given in 1974, is a frontal attack and a marvel of precision, in which the steps follow each other in an absolutely natural order, without surprises. Those who heard his lectures had the impression, day after day, that nothing new was happening–whereas every lecture by Grothendieck introduced a whole new world of concepts, each more general than the one before–but on the last day, everything was in place and victory was assured. Deligne knocked down the obstacles one after the other, but each one of them was familiar in style. I think that this opposition of methods, or rather of temperament, is the true reason behind the personal conflict which developed between the two of them. I also think that the fact that "John, the disciple that Jesus loved" wrote the last Gospel by himself partly explains Grothendieck's furious exile that Grothendieck has imposed upon himself.

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We now arrive at the very heart of Grothendieck's mathematical method, his unifying vision. Of the twelve grand ideas of which he was justly proud, he sets three above all the others: he gives them in the form of a progression:

$$\text{SCHEME} \to \text{TOPOS} \to \text{MOTIVE}$$

in an increasingly general direction. All of his scientific strategy was organized around a progression of increasingly general concepts. The image that occurs to me is that of a Buddhist temple that I visited in Vietnam in 1980. According to tradition, the altar was a series of rising steps, surmounted by a prone figure of Buddha – also traditional – a Buddha with a gigantic face, but whose features were actually those of a sage which local tradition described as the Vietnamese Montaigne of the 11th century, if one likes. When one follows Grothendieck's work throughout its development, one has exactly this impression of rising step by step towards perfection. The face of Buddha is at the top, a human, not a symbolic face, a true portrait and not a traditional representation.

Before explaining the meaning of the trilogy displayed above, it is important to talk about Grothendieck's stylistic qualities. He was a master at naming, and he used that ability as one of his main intellectual strategies. He had a particular talent for naming things before possessing and conquering them, and many of his terminological choices are quite remarkable. But also in this, his personal experience was unusual. His mother tongue was German, and he only spoke German with his mother, during the many years that he lived with her in symbiotic closeness, until her death. When I met him, around 1953, I felt when I spoke with him that he was thinking in German–and my lotharingian ear heard this correctly.

He had a remarkable sense of æsthetics. Yet I never understood his attraction to ugly women. I also cannot understand why he always lived in frightful homes: he worked at night, in general in a horrible room with the plaster falling off the walls, and turning his back to the window (seeking what secret humiliation?) And yet, when he sought mental images to explain his scientific ideas, he spoke of "the beautiful perfect manor", the "lovely inherited castle", all these allegories about beautiful homes. He even described himself as a builder. All these images are remarkably fitting and suitable. He may have continued to think for many years in German, but he certainly acquired a real sense of the French language, and his bilingualism enabled him to play on Germanic words. In French, he has a fantastically varied use of language ranging from the most familiar to the most elaborate, with an absolutely extraordinary sense of words.

His strategy, then, was to *name*. That is where I took the title of this article: "A country of which only the name is known", because that was truly his way of going about things. "Motives" represented the final step for him, the one that he did not reach, although he had successfully passed the two intermediate steps of schemes and toposes.

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It is out of the question to give here a technical introduction to the notion of a *scheme*. The term itself is due to Chevalley, in a more restrictive sense than Grothendieck's (see note [8]). In his *Foundations of Algebraic Geometry*, André Weil had extended to abstract algebraic geometry (i.e., over an arbitrary base field, not necessarily the real or complex numbers) the method of gluing via local charts that his teacher Élie Cartan had used in differential geometry (following Gauss and Darboux). But Weil's method was not intrinsic, and Chevalley had asked himself what was invariant in a variety in the sense of Weil – a question characteristic of Chevalley's style. The answer, inspired by previous works of Zariski, was simple and elegant: the scheme of an algebraic variety is the collection of local rings of the subvarieties, inside the rational function field. No mention of explicit topology, unlike Serre, who introduced his algebraic varieties using Zariski topologies and sheaves at just about this time. Each of the two approaches had its own advantages, but also its limits:

– Serre needed an algebraically closed base field;

– Chevalley needed to work only with irreducible varieties.

In both cases, the two fundamental problems of products of varieties and base change could only be approached indirectly. Chevalley's point of view was better adapted to future extensions to arithmetic, as Nagata discovered early on.

Galois was certainly the first person to notice the polarity between equations and their solutions. One must distinguish between the *domain*, in which the coefficients of the algebraic equation are chosen (the "constants") and the *domain* in which the solutions must be sought. Weil kept this distinction between the "field of definition" of a variety and the "universal domain", but he was not very explicit about whether the field of definition had an intrinsic meaning, obsessed as he was with his ideas of specialization. For Serre, there was to be only one domain (which was necessarily algebraically closed), which is satisfying for "geometric" problems, but masks a number of interesting questions. For Chevalley (following Zariski), the central object is the rational function field, with its field of definition appearing as the field of constants, and the universal domain is practically eliminated.

Grothendieck created a synthesis of all these ideas, essentially based on the conceptual presentation of Zariski-Chevalley-Nagata. Schemes, thus, are a way of encoding systems of equations, and the transformations they can undergo; ideal theory, developed at the beginning of the century by Macaulay and Krull, had already had some of the same ambitions, and we owe it a large number of technical results.

The way in which Grothendieck presented the Galois problem is as follows. A scheme is an "absolute" object, say X, and the choice of a field of constants (or a field of definition) corresponds to the choice of another scheme S and a morphism[28] π_X from X to S. In the theory of schemes, a commutative ideal is identified with a scheme, its spectrum[29], but to a homomorphism from the ring A to the ring B, there corresponds a morphism in the other direction from the spectrum of B to the spectrum of A. Moreover, the spectrum of a field has only a single underlying point (but there are many "different" points of this type); consequently, giving the field of definition as being included in the universal domain corresponds to giving a scheme morphism π_T from T to S. A solution of the "system of equations" X, with the "domain of constants" S, with values in the "universal domain" T, corresponds to a morphism φ from T to X such that π_T is the composition of φ and π_X, given symbolically as:

$$\begin{tikzcd} T \arrow[rr, "\varphi"] \arrow[dr, "\pi_T"'] & & X \arrow[dl, "\pi_X"] \\ & S & \end{tikzcd}$$

Admirable simplicity – and a very fruitful point of view – but a complete change of paradigm! The central point of view of "modern" mathematics is based on the central role of sets. Once one has accepted the existence of sets (simple "classes" or "collections"), and the constructions one can make with them (of which the most important is to be able to consider the subsets of a set as elements of a new set), every mathematical object is a set, and coincides with the set of its points[30]. Transformations are in principle transformations of points[31]. In the various forms of geometry (differential, metric, affine, algebraic), the central object is the variety[32], considered as a set of points. Already in the 19th

[28] From the start, this is based on the philosophy of categories: one defines the *category of schemes*, with its objects (schemes) and its transformations (morphisms); a morphism f links two schemes X and Y, which is symbolized by $X \xrightarrow{f} Y$.

[29] It was Gelfand's fundamental idea to associate a normed commutative algebra to a space. Grothendieck recalled his initial approach to functional analysis, exactly at the time, following 1945, when Gelfand's theory had come to occupy a central position. The term "spectrum" comes directly from Gelfand.

[30] This set must be "structured", which is done by using a set theoretic version of Russell's theory of types.

[31] But the possibility of considering, say, lines (or circles) in space as points of a new space makes it possible to incorporate the geometry of transformations of points into lines (or circles).

[32] In the etymological sense: "domain of variation".

century, mathematicians became used to distinguishing the real points from the complex points of curves or surfaces defined by polynomial equations. Even more: in the study of Diophantine equations, one considers a system of equations $f_1 = \cdots = f_m = 0$ in unknowns $x_1, \ldots, x_n$, where the polynomials $f_1, \ldots, f_m$ have coefficients which are integers. The study of these equations led people to distinguish real and complex solutions, but also integer or rational solutions; one can also consider a less orthodox kind, such as solutions in a Galois field (for example, the integers modulo a prime number p), or even, following Kummer and Hensel, a p-adic field. It was already common usage to look for the solutions of an equation considered more or less simultaneously everywhere. For Grothendieck, *the scheme is the internal mechanism, the matrix*[33], *which generates the points of the space*: the diagram above expresses this, by saying that φ is a T-point of the S-scheme X, and this for every S-scheme T.

In a recent article (see note [2]), I studied the problem of the geometric point in a very mathematical manner, and I will not repeat that analysis here. Let us simply say that the purely mathematical analysis, by Gelfand and then by Grothendieck, of the notion of a point, has recently crossed paths with a fundamental reflection in mathematical physics, about the status of the point in quantum physics. The most systematic expression of this last reflection is Alain Connes' "non-commutative geometry". The synthesis is far from complete. The slowly emerging close relationship between the *Grothendieck-Teichmüller group*[34] on the one hand and the *renormalization group* from quantum field theory[35] on the other is surely only the first manifestation of a symmetry group of the fundamental constants of physics – a kind of cosmic Galois group! Grothendieck had not predicted this development, and surely would not even have wished it, because of his prejudices against physics (essentially due to his violent rejection of the military-industrial complex). It is possible that these connections could have been investigated earlier if the constraints of the Soviet system had not put the brakes on the transmission of ideas across the iron curtain.

Somewhere in *Récoltes et Semailles*, Grothendieck compares himself to Einstein for his contribution to the problem of space. He is right, and his contribution is of the same depth as Einstein's[36]. Einstein and Grothendieck both deepened our vision of space, so that this is no longer an empty receptacle for phenomena, a neutral stage, but the main actor in the life of the world and the history of the Universe. This distant descendant of Descartes' theory of vortices is the principal motor of our comprehension of the physical world at the dawn of a new century.

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[33] I am using the word "matrix" here in its usual sense, not in the mathematical sense of a table of numbers.

[34] Thus baptized by Drinfeld, who is one of the mathematicians who penetrated the deepest into the above-mentioned *Esquisse d'un Programme* by Grothendieck

[35] Above all in the recent reformulation due to Connes and Kreimer.

[36] We should not forget Einstein's personal investment in the struggle against the military, from a political viewpoint quite close to Grothendieck's!

Let us now examine *toposes*[37]. We saw that the geometry of schemes is a geometry with a plethora of points, at least with the very generalized notion of point shown in the diagram above. Toposes, on the contrary, realize a *geometry without points.* The idea of a geometry without points is not new: in fact, it is the oldest one. From the Euclidean point of view, one considered geometric figures of which some were points, but there were also lines, planes, and circles; it was only in modern times, after the successes of set theory, that we adopted the habit of considering every component of a geometric figure as a set of points. Nowadays, a line is the set of its points: it is not a primitive object, but a composed one. However, nothing prevents one from proposing an axiomatic framework for geometry in which points, lines, planes and so forth would all be equal players, such as Birkhoff's axiomatic system for projective geometry, in which the primitive notion is that of a "plate" (a generalisation of lines, planes etc.) and the fundamental relation is that of containment: the point is in the line, the line is in the plane, etc. Mathematically, one considers a set of partially ordered sets called lattices[38], and a geometry corresponds to one of these lattices.

In the geometry of a topological space, and particularly in the use of sheaves, the lattice of the open sets plays a major role, and points are relatively secondary. Thus, one could replace a topological space by the lattice of its open sets without losing much, and this idea was considered at various times. But Grothendieck's originality was to pick up Riemann's idea that multivalued functions actually live, not on open sets of the complex plane, but on spread-out Riemann surfaces. The spread-out Riemann surfaces project down to each other and thus form the objects of a category. Now, a lattice is just a special case of a category; one in which there is at most one transformation between two given objects. Grothendieck proposed to replace the lattice of open sets by the category of spread-out open sets. Adapted to algebraic geometry, this idea solves a fundamental difficulty linked to the absence of an implicit function theorem for algebraic functions. This is how he introduced the "etale site" associated to a scheme. Sheaves can be considered as particular functors on the lattice of open sets (viewed as a category), and can thus be generalized to etale sheaves, which are particular functors of the etale site.

Grothendieck produced a number of variations on this theme, with remarkable success, in various problems of geometric construction (for example,the "moduli" problem for algebraic curves). His greatest successs was the possibility of defining the cohomological theory he needed to attack the Weil conjectures: it is called the ℓ-adic etale cohomology of schemes.

But there is one more level in this movement towards abstraction. Consider the progression:

$$\text{SCHEME} \to \text{ETALE SITE} \to \text{ETALE SHEAVES}$$

[37] Certain purists would like the plural to read "topoi" as in classical Greek. I will follow Grothendieck, writing "topos" and "toposes".

[38] The hypotheses needed here are the existence of a largest and smallest element (the empty set and the universal set), and of intersections and joins of two plates. In the last twenty years, this point of view was redeveloped under the name of "matroid" or "combinatorial geometry" (mainly by Rota and Crapo).

Grothendieck realized that one can place oneself directly at the last and highest level, and that all the geometric properties of a scheme are encoded in the *category of etale sheaves*. This category belongs to a particular type of categories that he called "toposes".

This, then, is the final act of the play. It was typical of the wild generosity of Grothendieck's ideas, and also of the lightheartedness with which he abandoned his (mathematical) children. Our hero had noticed that the sheaves on a given space formed a category which appeared to have all the same properties as "the" category of sets. But after the undecidability results of Gödel and Cohen in set theory, we know that there is not just *one* category of sets, but *many non-equivalent models* of set theory (in the logical sense of "model"). It was thus natural to explore the relations between toposes and models of set theory. Grothendieck was as ignorant, and perhaps as contemptuous of Logic as his Master, Bourbaki, and his attitude towards mathematical physics was no different. It was for others (above all Bénabou, Lawvere and Tierney) to resolve the puzzle: toposes are exactly models of set theory, but in a very particular logic, called 'intuitionist', in which the principle of excluded middle is not valid. It is remarkable that this logic was invented by a famous topologist, Brouwer, and that with a little perspective, it arises very naturally by virtue of the fact that the interior of the closure of an open set is not equal to it.[39]

But the invention of topolses gives an unheard-of freedom to the mathematical game, and makes it possible to break the yoke of the "only" set theory. To play a familiar mathematical piece in the new decor of a somewhat exotic topos can bring new surprises, and reveal new accents in well-known verses, and sometimes this new representation actually brings forth a mathematical treasure. From a more general point of view, a topos carries its own logic within it[40], and thus defines a kind of modal logic, or rather a *hic* and *nunc* logic, a spatio-temporal logic in which the truth value of an assertion can depend on the place and time[41].

From a more technical point of view, Peter Freyd successfully applied methods from toposes to simplify Cohen's "forcing" method, and his proof of the undecidability of the continuum hypothesis. It would be just as desirable to use methods from toposes together with recent results from model theory concerning the Mordell-Lang conjecture[42]

One understands better now why Grothendieck considered the notion of toposes as central, while the more general concept of categories was nothing more for him than a tool.

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[39] A topological version of the fact that the double negation of a property is not necessarily equivalent to it (in intuitionist logic), in violation of the hypothesis of excluded-middle.

[40] In technical terms, in every topos, the set of subobjects of the final object is a Heyting lattice, an intuitionist version of an algebra of propositions (Boolean lattices being the "classical" logical version.

[41] On my suggestion, the lawyer Mireille Delmas-Marty and the mathematician Jean Bénabou met to examine the possibility of founding the theoretical basis of federal law (of the European type) on the theory of toposes. I do not believe these efforts were actually successful, but the idea would be worth revisiting.

[42] On this subject, see a recent lecture by Elisabeth Bouscaren at the Bourbaki Seminar (March 2000, exposé 870).

It remains for us to make some remarks on the subject of *motives*. The image that Grothendieck gave himself was that of a rocky coastline at night, illuminated by a lighthouse. The beam from the lighthouse turns, illuminating first one part of the coast and then another. In a similar manner, the various known cohomological theories, of which several that he invented himself, are what we see, and it is necessary to go back to the source and build the lighthouse which will unify the representation of the entire coastline. In a certain sense, the scientific strategy is the inverse of the one he used in scheme theory. In the diagrammatic representation given above, the S-scheme X was given, and from there, one could realize its diverse incarnations: for every S-scheme T, one can construct the set of T-points of X. Here, the starting place is unknown, and only the various incarnations are in our possession: is this a theological image?

Grothendieck published nothing on this subject, contenting himself with a few remarks. I believe that Manin was the first to provide a real contribution, and then there was a long silence. Over the last few years, there has been an increase of activity and the program has become much more precise. The most ambitious contribution has been Voevodsky's: he contructs a category of objects, called motives, which is the locus of geometric invariants, and each scheme defines a particular motive. But in such a category, "pieces of objects" can migrate; the image of a genetic inheritance migrating through different beings is a good one. That this is possible follows from the definition of "weight" given by Deligne, which was the main ingredient in his proof of the Weil conjectures.

The tool created by Voevodsky undoubtedly corresponds to what Grothendieck expected, but it is going to be difficult to use. Good tools should be easy to use. Thus, the progress that has been made has been accomplished by restricting ambition to less general notions, called "mixed Hodge structures" or "mixed Tate motives", each of which is the expression of a fundamental group of symmetries, like the Grothendieck-Teichmüller group mentioned above. In fact, even within this limited scope, there is already an enormous amount of work to be done, and inestimable treasures to be unearthed. Grothendieck complained that all this was too economical, too reasonable, and from the heights of his visionary attitude, he heaped reproaches on the workers. But it seems to me that in the presence of mathematical visionaries like Grothendieck – or Langlands – who formulated wildly ambitious but sometimes imprecise programs of research, the right scientific strategy consists in isolating one piece which is sufficiently precise and restricted that one can actually work with it, and sufficiently vast to yield interesting results. The worker's philosophy?

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Anatomy of an author

I will not venture upon a diagnosis of our patient, not being really competent to do so; but I will make some remarks, guided by sympathy. What is striking about Grothendieck is the expression of *suffering*: suffering because of having left an unfinished work, and the feeling of having been betrayed by his collaborators and followers. In a moment of true lucidity, he said something like: "I was the only person to have the inspiration; and what I transmitted to those around me was the task. I had workers around me, but none of them really had the inspiration!" The comment is deep and true, but it doesn't answer the question of why he deliberately closed the source of that inspiration! From what we know of his life today, he is subject to periodic crises of depression. It seems to me that his capacity for scientific creation was the best antidote to depression, and that the immersion in a living scientific milieu (Bourbaki and the IHES) helped this creation to take place by giving it a collective dimension; contrarily, in the relative scientific desert of Montpellier, and even more in his fiercely defended retreat, the isolation and the lack of minds at his own level with whom to discuss and compare himself no longer protect him from these eruptions of suffering.

To remain on a more secure terrain, I would like to say something about the religious aspect of his life. Whether it is permanent or deep emerges from what he says. He has had experiences of visual and auditive hallucination he has described divine apparitions and speaks of canticles sung by two simultaneous voices, his own and that of God. It was following a series of these hallucinations – or apparitions – that he sent out a public eschatological message, which received no answer!

What were his antecedents? I have already noted that his father was born in a Hassidic community in the Ukraine, there where for the glory of God, hermits used to have themselves walled into towers with nothing but a tiny opening to the outside through which food might come in the form of alms from the faithful. But Grothendieck was never attached to Judaism, in any of its established forms. He felt closer to the Buddhist tradition; I don't know who first introduced him to that way of thinking, but I have already mentioned that visitor who unintentionally provoked his trial. At the end of the 60s, Grothendieck visited Vietnam which was at that time the butt of American bombardments, and he had a long-term liaison in France with a Vietnamese student (officially a good Communist, but...) One of his main obsessions was about food, and at times he practiced an extreme form of vegetarianism. That is certainly an area where the Judaic and Buddhist traditions meet.

His personal Trinity was composed of God the Father, the goddess-mother, and the Devil. He calls the first "le bon Dieu": I don't know why he uses that term which corresponds to a somewhat outmoded popular usage, but it seems clear that he is not referring to Buddha, but rather to the image of the absent father (and in any case, in Buddhist orthodoxy, Buddha is not God!) The main character is the goddess-mother, whom he describes somewhere as a seductive female figure named Flora. The goddess-mother is present in many religions (including officially monotheistic Christianity), but a fairly recent phenomenon is the development in Japan and in Vietnam of the cult of Kannon (or Kan-Eum, or Lady of Mercy[43]. The relation between this worship of Grothendieck and his own mother is obvious. He lived in close symbiosis with her as she became progressively weaker and more disabled after her experiences in French detention camps, during the

[43] During a recent trip to Vietnam, I took note of certain curious phenomena of imitation between the Virgin Mary of the (still very numerous) Catholics and the Kannon of the Buddhists, not counting when it was adopted on top of everything else by the Communist regime (for example in a votive stela installed at the place of the last tsunami in Hué). After all, sculptors can work for various masters. The number of new sanctuaries dedicated to Kannon is amazing!

nearly two decades from his arrival in Paris in 1939 to her death in 1957. His name is her name[44], he dedicated his thesis to her and together they shared her native language of German. According to his own testimony, his wild passion for women had to wait until his mother's death to fully emerge, from the end of the 1950's to the end of the 1980's.

The most worrisome symptom is his obsession with the Devil. According to his most recent visitors, he, who never theologized his religion, has plunged into the writing of a sermon on the Devil's action in our world (he always was obsessional about writing!) His "catastrophism" is by no means new, and his concerns about the terrors of global nuclear war and pollution came at the right time in the 1970's. More recently, there was the incident mentioned earlier where, like Paco Rabanne, he received a revelation of the date of the end of the world, and he made it quite clear that it was going to be the end of "everything", not just our little Earth. Out of charity, he communicated this date to 200 or 300 people drawn from his list of scientific correspondents, and exhorted them to repentance before the final explosion, for afterwards, there would remain but a few chosen. That letter, which received no response, was followed by a gloomy retraction after the fatidic date.

I will only describe two episodes to illustrate to what point he has become distanced from a rational and scientific point of view. About ten years ago, during a visit to Montpellier, he showed me the work of two of his students: a long enumeration, using colored pencils, of configurations of lines (the problem was serious). When I pointed out that a computer calculation would have been quicker and surer, he responded sadly that that suggestion showed me up as an envoy of the Devil (in his military-industrial version!) More recently, he plunged into a series of long reflections in order to understand how the 300,000 km/sec that divine harmonie would require for the speed of light had managed to become 298,779 km/sec by the corrupting influence of the Devil. He, whose mathematical work was so anchored around the notion of invariance and the naturalness of concepts refused to perceive the conventional nature of the metric system[45]. It isn't a question of error or of scientific ignorance: it is just an example of the other side of his own personal logic, that very same unstable logic which gave us his prodigious work.

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In the place of a conclusion

Mathematics sees itself as the most *objective* of all the sciences. At the very least, its *intersubjectivity* requires that the mathematical experience be as detached as possible from

[44] It was not without difficulty that I finally discovered the name of his father, which he never mentioned. I thank some Russian friends for having helped me with accomplishing this research, after *perestroika*.

[45] The creators of the metric system insisted on the *rational and natural* nature of their system. It took some time to really understand the degree to which convention played a role within it, and this is what gave rise to the permanent efforts to base our international system of units on a truly more natural foundation. But then, even the base 10 used in the decimal system is conventional!

the affect of the mathematician, in order to be communicated without distorsion, respecting its collective nature. The mathematical subject, conceived to be the mathematician subject present behind the creation, is required to disappear, and in practice, this disappearance is quite effective.

In this situation, Grothendieck represents an extremely special case. He, whose father was at the heart of every social combat for half a century, lived outside of the world, even much more so than the traditionally absent-minded professor. Even in his mathematical milieu, he wasn't quite a member of the family, and essentially pursued a kind of monolog, or rather, a dialog with mathematics...and God, as he did not separate the two things. His work is unique in that his fantasies and obsessions are not erased from it, but live within it, and it takes its life from them: at the same time as he gave us a strictly mathematical work, he also delivered to us, in a Freudian sense, what he believed to be its meaning.

His life was burned by the fire of the spirit, like that of Simone Weil, with whom he shares a strange kinship. He searched for a country, and for a name. Was it the Promised Land, that Judaea flowing with milk and honey, barely perceived beyond Mount Sinai, the accomplishment of a promise, the revelation of the whole and entire truth? I believe that the mythical land was the land of his father, that Jewish Ukraine that the sorrows of Eastern Europe rendered more inaccessible even than the Soviet Empire, already one of the most closed-off places in the world. The name – is that of the father.

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Alexander Grothendieck was (is) a genius of the first order, and a truly amazing spirit. Freeman Dyson once categorized mathematicians as being of roughly two types: birds and frogs. The latter group studies the fine details of the terrain; the former group soars high above and surveys the landscape. Grothendieck was the highest-soaring bird â€” to Grothendieck, a problem was not truly solved until it was viewed from the "right" general perspective, from which it could be solved effortlessly, from which it became in a sense obvious, from which it fit naturally into a larger conceptual