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About $k-$ Fibonacci Numbers and their Associated Numbers

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Abstract

In this paper we define the associated $k-$Fibonacci numbers and we give a combinatorial interpretation for them.

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1 Introduction

Ever since Fibonacci modelled the growth of a rabbit population in his book Liber Abaci, the Fibonacci numbers and $\phi = \frac{1+\sqrt{5}}{2}$, a special constant related to them, have been applied countless times in art, science and architecture.

Besides the usual Fibonacci numbers many kinds of generalizations of these numbers have been presented in the literature(e.g. see [1-5]). The well-known Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined as

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2$$

where F_n denotes the $n-$th Fibonacci number.
For any positive real number k, the $k-$Fibonacci sequence, say $\{F_{n,k}\}_{n=0}^{\infty}$ is defined recurrently by

$$F_{n+1,k} = kF_{n,k} + F_{n-1,k}$$

with initial conditions

$$F_{0,k} = 0 \text{ and } F_{1,k} = 1.$$

In [2], these general $k-$Fibonacci numbers$\{F_{n,k}\}_{n=0}^{\infty}$ were found by studying the recursive application of two geometrical transformations used in the well known four-triangle longest-edge(4TLE) partition. Many properties of these numbers are obtained directly from elementary matrix algebra. In [3], many properties of these numbers are deduced and related with the so-called Pascal 2-triangle. In [4], authors defined $k-$Fibonacci hyperbolic functions similar to hyperbolic functions and Fibonacci hyperbolic functions. They deduced some properties of k-Fibonacci hyperbolic functions related with the analogous identities for the $k-$Fibonacci numbers. Finally, authors studied 3-dimensional $k-$Fibonacci spirals with a geometric point of view in [5]. Several properties of these new $k-$Fibonacci hyperbolic functions are studied in an easy way. In the 19th century, the French mathematician Binet devised two remarkable analytical formulas for the Fibonacci and Lucas numbers [6]. Some identities for the $k-$Fibonacci numbers may be found in [7].

Members of the k-Fibonacci sequence $\{F_{n,k}\}_{n=0}^{\infty}$ will be called $k-$Fibonacci numbers. Some of them are:

n	$F_{n,k}$
0	0
1	1
2	k
3	k^2+1
4	k^3+2k
5	k^4+3k^2+1
6	k^5+4k^3+3k
7	$k^6+5k^4+6k^2+1$
8	$k^7+6k^5+10k^3+4k$
9	$k^8+7k^6+15k^4+10k^2+1$
10	$k^9+8k^7+21k^5+20k^3+5k$
11	$k^{10}+9k^8+28k^6+35k^4+15k^2+1$
12	$k^{11}+10k^9+36k^7+56k^5+35k^3+6k$
13	$k^{12}+11k^{10}+45k^8+84k^6+70k^4+21k^2+1$
14	$k^{13}+12k^{11}+55k^9+120k^7+126k^5+56k^3+7k$
15	$k^{14}+13k^{12}+66k^{10}+165k^8+210k^6+126k^4+28k^2+1$
16	$k^{15}+14k^{13}+78k^{11}+220k^9+330k^7+252k^5+84k^3+8k$
17	$k^{16}+15k^{14}+91k^{12}+286k^{10}+495k^8+462k^6+210k^4+36k^2+1$

Particular cases of the previous definition are:

- If $k=1$, the classic Fibonacci sequence is obtained:
$F_0=0$, $F_1=1$, and $F_{n+1}=F_n+F_{n-1}$ for $n\geq 1$:
$\{F_n\}_{n=0}^{\infty}=\{0,1,1,2,3,5,8,...\}$.
- If $k=2$, the classic Pell sequence appears:
$P_0=0$, $P_1=1$, and $P_{n+1}=2P_n+P_{n-1}$ for $n\geq 1$:

$\{P_n\}_{n=0}^{\infty} = \{0, 1, 2, 5, 12, 29, 70, ...\}$.

• If $k = 3$, the following sequence appears:

$H_0 = 0$, $H_1 = 1$ and $H_{n+1} = 3H_n + H_{n-1}$ for $n \geq 1$:

$\{H_n\}_{n=0}^{\infty} = \{0, 1, 3, 10, 33, 109, ...\}$.

2 The associated $k-$Fibonacci numbers and their combinatorial interpretation

We define the sequence $\{A_{n,k}\}_{n=0}^{\infty}$ associated to$\{F_{n,k}\}_{n=0}^{\infty}$ as

$$A_{0,k} = 1 \text{ and } A_{n,k} = F_{n,k} + F_{n-1,k} \text{ for } n = 1, 2, 3, ..$$

Observe that for $n = 1, 2, 3, ...$ the expression $A_{n,k}$ is the sum of the two consecutive $k-$Fibonacci numbers $F_{n,k}$ and its predecessor $F_{n-1,k}$. The members of the sequence $\{A_{n,k}\}_{n=0}^{\infty}$ will be called *associated* $k-$Fibonacci numbers. An equivalent definition for the sequence $\{A_{n,k}\}_{n=0}^{\infty}$ is

$$A_{n,k} = \begin{cases} 1 & \text{if} \quad n = 0 \\ 1 & \text{if} \quad n = 1 \\ (k+1)F_{n-1,k} + F_{n-2,k} & \text{if} \quad n \geq 2 \end{cases}$$

Observe that

$$\begin{aligned} A_{n,k} &= F_{n,k} + F_{n-1,k} \\ &= kF_{n-1,k} + F_{n-2,k} + kF_{n-2,k} + F_{n-3,k} \\ &= k(F_{n-1,k} + F_{n-2,k}) + (F_{n-2,k} + F_{n-3,k}) \\ &= kA_{n-1,k} + A_{n-2,k}. \end{aligned}$$

This allows to define recursively the sequence of associated $k-$Fibonacci numbers as follows :

$$A_{n,k} = \begin{cases} 1 & \text{if} \quad n = 0 \\ 1 & \text{if} \quad n = 1 \\ kA_{n-1,k} + A_{n-2,k} & \text{if} \quad n \geq 2 \end{cases} \tag{1}$$

Following table shows some of the associated $k-$Fibonacci numbers

n	$A_{n,k}$
0	1
1	1
2	$k+1$
3	k^2+k+1
4	k^3+k^2+2k+1
5	$k^4+k^3+3k^2+2k+1$
6	$k^5+k^4+4k^3+3k^2+3k+1$
7	$k^6+k^5+5k^4+4k^3+6k^2+3k+1$
8	$k^7+k^6+6k^5+5k^4+10k^3+6k^2+4k+1$
9	$k^8+k^7+7k^6+6k^5+15k^4+10k^3+10k^2+4k+1$
10	$k^9+k^8+8k^7+7k^6+21k^5+15k^4+20k^3+10k^2+5k+1$
11	$k^{10}+k^9+9k^8+8k^7+28k^6+21k^5+35k^4+20k^3+15k^2+5k+1$

For a fixed integer $k \geq 1$ we define

$$f_{0,k} = 1 \text{ and } f_{1,k} = 1 \text{ for } k = 1, 2, 3, ... \tag{2}$$

Let us consider the set $S = \{0, \alpha_1, \alpha_2, ..., \alpha_k\}$ whose elements are 0 and k pairwise distinct symbols α_1, α_2, ..., α_k such that $\alpha_i \neq 0$ $(i = 1, 2, ..., k)$.
For any $n \geq 2$ let $f_{n,k}$ be the number of $(n-1)-$permutations of the set S with repetition and the restriction that no two equal symbols α_i are consecutive. We will call this restriction R. We have the following
Theorem. For any $k \geq 1$ and any $n \geq 2$,

$$f_{n,k} = kf_{n-1,k} + f_{n-2,k}.$$

Proof. We will prove this by induction on n. The theorem is valid for $n = 2$ since the number of $1-$permutations of the set $S = \{0, \alpha_1, \alpha_2, ..., \alpha_k\}$ with repetition satisfying restriction R equals $k + 1$. They are

$$0, \alpha_1, \alpha_2, ..., \alpha_k$$

Suppose that theorem holds for any n with $2 \leq n \leq m$. The $m-$permutations of the set $S = \{0, \alpha_1, \alpha_2, ..., \alpha_k\}$ satysfying restriction R are divided into two mutually disjoint classes: the class I of the permutations ending with 00 and the class II of permutationsfor which the two last elements are distinct symbols of the set S.
Given any permutation of the class I we obtain a $(m-2)$-permutation satisfying restriction R if we delete the last two zeros in it. Conversely, if we have a $(m-2)$-permutation satisfying restriction R we obtain a permutation of the class I by adding two zeros at the end of it. By the inductive hypothesis, the class I has $f_{m-1,k}$ elements.

On the other hand, if $\beta_1\beta_2\cdots\beta_{m-1}$ is any $(m-1)$-permutation satisfying restriction R we obtain k permutations of class II by adding at the end of it any element in the set $S_0 = \{0, \alpha_1, \alpha_2, ..., \alpha_k\}\diagdown\{\beta_{m-1}\}$. The result is the $m-$permutation $\beta_1\beta_2\cdots\beta_{m-2}\beta_{m-1}\gamma$. Conversely, if we have a permutation $\beta_1\beta_2\cdots\beta_{m-2}\beta_{m-1}\beta_m$ of class II we obtain a $(m-1)$-permutation satisfying restriction R by deleting any element on it. In view of the inductive hypothesis, the class II contains $kf_{m,k}$ elements. Thus

$$f_{m+1,k} = kf_{m,k} + f_{m-1,k}$$

Theorem has been proved.

From previous theorem and taking into account (1) and (2) we conclude that

$$A_{n,k} = f_{n,k} \text{ for all } n \geq 1 \text{ and } k \geq 1.$$

We obtained an interpretation of the associated $k-$Fibonacci numbers in terms of permutations. Since a permutation generates a word we may interpret these numbers in terms of words of given length. As an illustration, Table 1 shows some of these permutations for $k = 1$. Observe that $f_{n,1}$ is the number of $(n-1)-$permutations of the set $S = \{0, 1\}$ with repetition and the restriction that no two 1 are consecutive $(n = 2, 3, 4, 5, ...)$.

0 1	00 01 10	000 001 010 100 101
$f_{2,1} = 2 = f_{1,1} + f_{0,1}$	$f_{3,1} = 3 = f_{2,1} + f_{1,1}$	$f_{4,1} = 5 = f_{3,1} + f_{2,1}$

0000 0001 0010 0100 0101 1000 1001 010

$f_{5,1} = 8 = f_{4,1} + f_{3,1}$

00000 00001 00010 00100 00101 01000 01001
01010 10000 10001 10010 10100 10101

$f_{6,1} = 13 = f_{5,1} + f_{4,1}$

000000 000001 000010 000100 000101 001000 001001
001010 010000 010001 010010 010100 010101 100000
100001 100010 100100 100101 101000 101001 101010

$f_{7,1} = 21 = f_{6,1} + f_{5,1}$

Table 1. Number of permutations $f_{n,k}$ for $k = 1$ and $n = 2, 3, 4, 5, 6, 7$.

From Table 1 we see that

$$f_{n,1} = f_{n-1,1} + f_{n-2,1} \ (n = 2, 3, 4, 5, 6, 7)$$

where

$$f_{0,1} = 1 \text{ and } f_{1,1} = 1.$$

In other words, $f_{n,1}$ are the well known Fibonacci numbers. They are in essence the same associated Fibonacci numbers.
On the other hand, on Table 2 we may observe some permutations associated to the numbers $f_{n,k}$ for $k = 2$. Observe that $f_{n,2}$ is the number of $(n-1)-$permutations of the set $S = \{0, 1, 2\}$ with repetition and the restriction that no two 1 and no two 2 are consecutive $(n = 2, 3, 4, 5, ...)$.

0 1 2

$f_{2,2} = 3 = 2f_{1,2} + f_{0,2}$

00 01 02 10 12 20 21

$f_{3,2} = 7 = 2f_{2,2} + f_{1,2}$

000	001	002	010	012	020
021	100	101	102	120	121
200	201	202	210	212	

$f_{4,2} = 17 = 2f_{3,2} + f_{2,2}$

0000	0001	0002	0010	0012	0020	0021	0100	0101
0102	0120	0121	0200	0201	0202	0210	0212	1000
1001	1002	1010	1012	1020	1021	1200	1201	1202
1210	1212	2000	2001	2002	2010	2012	2020	2021
2100	2101	2102	2120	2121				

$f_{5,2} = 41 = 2f_{4,2} + f_{3,2}$

00000	00001	00002	00010	00012	00020	00021	00100	00101	00102	00120
00121	00200	00201	00202	00210	00212	01000	01001	01002	01010	01012
01020	01021	01200	01201	01202	01210	01212	02000	02001	02002	02010
02012	02020	02021	02100	02101	02102	02120	02121	10000	10001	10002
10010	10012	10020	10021	10100	10101	10102	10120	10121	10200	10201
10202	10210	10212	12000	12001	12002	12010	12012	12020	12021	12100
12101	12102	12120	12121	20000	20001	20002	20010	20012	20020	20021
20100	20101	20102	20120	20121	20200	20201	20202	20210	20212	21000
21001	21002	21010	21012	21020	21021	21200	21201	21202	21210	21212

$f_{6,2} = 99 = 2f_{5,2} + f_{4,2}$

Table 2. Number of permutations $f_{n,k}$ for $k = 2$ and $n = 2, 3, 4, 5, 6$.

From Table 2 we see that

$$f_{n,2} = 2f_{n-1,2} + f_{n-2,2} \ (n = 2, 3, 4, 5, 6)$$

where

$$f_{0,2} = 1 \text{ and } f_{1,2} = 1.$$

In other words, $f_{n,2}$ are the associated Pell numbers.

3 References

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International Conference on Fibonacci Numbers and Their Applications. <3rd : 1988 Pisa, Italy). Applications of Fibonacci numbers proceedings of the Third.Â On fibonacci numbers and their applications. A newspaper article at Pisa, Italy, with a prominent headline: "CONVEGNO PARLANO I MATEMATICI L'INCONTRO IN OMMAGIO A FIBONACCI" heralded our Third International Conference on Fibonacci Numbers and Their Applications which was held in Pisa, Italy, July 25th-29th, 1988.Â Meanwhile, in 1963, Hoggatt and his associates founded The Fibonacci Association and began publishing The Fibonacci Quarterly. They also organized a Fibona.cci Conference in California, U.S.A., each year for almost sixteen years until 1979.