ON A MEASURE OF DEPENDENCE FOR EXTREME VALUE COPULAS

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Abstract. Copulas provide a flexible tool for analysing a possibly nonlinear dependence relation among random variables. Here we consider the measure of dependence for the extreme value copulas, which are obtained through the limting procedure and are related to the extreme events. It is known that the Kendall's tau and the Spearman's rho are served as such characteristic quantities of measure of concordance. We extend and generalize these measures of association in the case of extreme value copulas.

1. Introduction

There has been much interest in the structure of dependence relations among risk factors both from theoretical and practical modeling viewpoint. To analytically measure such dependence relations, several characteristic quantities have been introduced and widely employed, which include, to name a few, the population version of Kendall's tau (τ) and/or Spearman's rho (ρ).

Copulas, on the other hand, are well recognized to provide a flexible tool for understanding the dependence relation among random variables (see for example [3]). It is therefore also known that the above τ and ρ are formulated in terms of copulas.

In this paper, we introduce a kind of generalized measures of dependence, which are expressed through the form of copulas. In particular, we focus our study on a special class of extreme value copulas (see [4]).

The so-called extreme value copulas naturally arizes in the field of extreme events and is defined by certain limiting procedure. In the bivariate case, on which our attention is paid, the extreme value copula is given through specific class of functions, known as Pickands dependence function. With these concepts in hand, we propose a generalization of dependence measures for extreme value copulas, which, we hope, may be served as a fitting criterion of modeling.

This paper extends the contents of [5], which is supervised by the present authors. The current version is based on new studies as well as necessary corrections of the previous one. We refer also to our related work [6].

The paper is organized as follows: Section 2 gives basic definition and properties of copulas. Our main results is addressed in Section 3. Examples of some computation are presented in Section 4. Section 5 concludes with discussions.

2. Copulas and the mesures of dependence

We begin with recalling the definition of copulas in the case of bivariate joint distribution.
Definition. A function C defined on $\mathbb{I}^2 := [0,1] \times [0,1]$ and valued in $\mathbb{I} := [0,1]$ is said to be a copula if the following conditions are satisfied.

Key words: dependence relation, copulas, measure of dependence, extreme value copulas.

(i) For every $(u,v) \in \mathbb{I}^2$,

$$C(u,0) = C(0,v) = 0, \quad C(u,1) = u \quad \text{and} \quad C(1,v) = v. \tag{2.1}$$

(ii) For every $(u_i, v_i) \in \mathbb{I}^2$ $(i = 1,2)$ with $u_1 \le u_2$ and $v_1 \le v_2$,

$$C(u_1,v_1) - C(u_1,v_2) - C(u_2,v_1) + C(u_2,v_2) \ge 0. \tag{2.2}$$

The requirement (2.2) is referred to as *the* 2*-increasing condition.* It is noted that a copula is a continuous function by its definition.

The class of extreme value copulas, on which our main concern is placed, is defined as follows.

Definition 1. A copula C_* is called an extreme value copula if there exists a copula C such that

$$C_*(u,v) = \lim_{n\to\infty} (C(u^{1/n}, v^{1/n}))^n$$

for $u, v \in \mathbb{I}$. C is said to belong to the domain of attraction of C_*.

The extreme value copula is known to be chracterized by a single function, which is stated as follows.

Theorem 2. *A bivariate copula C_* is an extreme value copula if and only if*

$$C_*(u,v) = \exp\left(\log(uv) A\Big(\frac{\log v}{\log(uv)}\Big)\right), \tag{2.3}$$

where $A : \mathbb{I} \to [1/2, 1]$ is convex and verifies $\max\{t, 1-t\} \le A(t) \le 1$ for every $t \in \mathbb{I}$.

For the proof of above theorems, we refer for instance to a book by Nelsen [7]. The function A is called the Pickands dependence function.

The population version of Kendall's tau (τ) and Spearman's rho (ρ) are well known measures of dependence. It is also known that τ and ρ can be represented in terms of copulas. Precisely, let X and Y be continuous random variables whose copula is C. Then we have

$$\begin{aligned}
\tau_{X,Y} = \tau_C &= 4 \iint_{\mathbb{I}^2} C(u,v) dC(u,v) - 1 = 1 - 4 \iint_{\mathbb{I}^2} \frac{\partial C}{\partial u}(u,v) \frac{\partial C}{\partial v}(u,v)\, dudv, \\
&= \int_0^1 \frac{t(1-t)}{A(t)} dA'(t) \\
\rho_{X,Y} = \rho_C &= 12 \iint_{\mathbb{I}^2} uv dC(u,v) - 3 = 12 \iint_{\mathbb{I}^2} C(u,v)\, dudv - 3 \\
&= 12 \int_0^1 \frac{1}{(1 + A(t))^2} dt - 3.
\end{aligned}$$

In the next section, we discuss a kind of generalizations for above measures of association.

3. Extensions and generalizations

Now we want to extend and generalize the measures $\mathcal{M}(C)$ of dependence involving the copula C, which should be of the form

$$\mathcal{M}(C) = \iint_{\mathbb{I}^2} f(u, v, C(u,v)) dC(u,v), \tag{3.1}$$

where $f = f(u, v, C)$ is an appropriate smooth function nondecreasing in C, whose detailed assumptions should be clarified below. For example of this form, we note that the Kendall's tau is provided by

$$f(u, v, C) = 4C - 1$$

and the Spearman's rho is

$$f(u, v, C) = 12uv - 3.$$

Therefore we understand that the formula (3.1) is a natural generalization. We here remark that $\iint_{I^2} dC(u, v) = 1$.

To proceed, we have to recall that M. Scarsini [8] has formulated a set of axioms that a measure $\mathcal{M}(C)$ of concordance for every bivariate copula C should satisfy (see [2]):

(1) $-1 \le \mathcal{M}(C_1) \le \mathcal{M}(C_2) \le 1$ if $C_1 \le C_2$;
(2) $\mathcal{M}(C) = \mathcal{M}(C^T)$, where C^T is the transpose of C, namely, $C^T(u, v) := C(v, u)$;
(3) $\mathcal{M}(\Pi) = 0$, where $\Pi(u, v) := uv$ is the product copula;
(4) $\mathcal{M}(C^{\sigma_1}) = \mathcal{M}(C^{\sigma_2}) = -\mathcal{M}(C)$, for the symmetries σ_1, σ_2 of $\mathbb{I}^2$, that is, for example $C^{\sigma_1}(u, v) = v - C(1 - u, v)$;
(5) if $C_n \to C$ uniformly as $n \to \infty$, then $\lim_{n\to\infty} \mathcal{M}(C_n) = \mathcal{M}(C)$.

Concerning these axioms, we also refer to [1][9][10] for related topics. Moreover, as to the axiom (1), it customarily suffices to impose that

$$\mathcal{M}(W) = -1 \qquad \text{and} \qquad \mathcal{M}(M) = 1$$

for the boundary condition, where $W(u, v) := \max\{u + v - 1, 0\}$ and $M(u, v) := \min\{u, v\}$. This is due to the so-called Hoeffding-Fréchet bounds: $W(u, v) \le C(u, v) \le M(u, v)$ for every copula C.

However, the class $\mathcal{E}$ of extreme value copulas, with which we are concerned, is not necessarily invariant by symmetries. To be precise, it does not follow that $C \in \mathcal{E}$ implies $C^{\sigma_1} \in \mathcal{E}$ nor $C^{\sigma_2} \in \mathcal{E}$. We thus do not take into account the axiom (4) above in this study and we presume that our (3.1) should verify the set of axioms (1)(2)(3)(5). Even this set of axioms will make certain restriction on f.

Furthermore, we confine ourselves to the case $f = f(uv)$ for simplicity, and we wish to consider a generalized measure of concordance of the form

$$\mathcal{M}(C) = \iint_{\mathbb{I}^2} f(uv)dC(u, v). \tag{3.2}$$

Our main result now reads as follows.

Theorem 3. *Let $f = f(t)$ be a nondecreasing smooth function which satisfy*

(3.3)
$$\int_0^1 \Big(\frac{1}{t}\int_0^t f(s)ds\Big)dt = 0,$$

$$\iint_{\mathbb{I}^2} f(uv)dW(u, v) \ge -1, \quad \iint_{\mathbb{I}^2} f(uv)dM(u, v) \le 1, \quad \textit{where at least one equality holds.}$$

Then for the measure $\mathcal{M}(C)$ of concordance (3.2) with C being the extreme value copula (2.3) above, we have

$$\mathcal{M}(C) = \int_0^1 (f(s) - sf'(s))ds + \int_0^1 dt \int_0^1 (f'(s) + sf''(s))s^{A(t)} \log s \, ds. \tag{3.4}$$

Sketch of Proof. The proof is just an elementary computation. We make the change variables $t = \log v / \log(uv)$ and $(u, v) \to (u, t)$. Then we see that $v = u^{t/(1-t)}$ and

$$dudv = \frac{u^{\frac{t}{1-t}} \log u}{(1-t)^2} dudt.$$

The conditions (3.3) comes from the axioms (2) and (1). We may safely omit the details. □.

4. Examples

We here present examples to illustrate our method.

Example 1. If we take

$$f_n(t) = \frac{(2n+1)(n+1)^2}{n^2} t^n - \frac{2n+1}{n^2} \qquad (n = 1, 2, \cdots) \tag{4.1}$$

and compute the corresponding formula for (3.1). We infer that

$$\begin{aligned} \mathcal{M}(C) &= (2n+1)(n+1)^2 \iint_{\mathbb{I}^2} (uv)^{n-1} C(u,v) dudv - (2n+1) \\ &= (2n+1)(n+1)^2 \int_0^1 \frac{dt}{(n + A(t))^2} - (2n+1). \end{aligned}$$

If we take $A(t) = (t^\theta + (1-t)^\theta)^{1/\theta}$ with $\theta \geq 1$, where we recover the so-called Gumbel-Hougaard family of copulas, then we learn that

$$\mathcal{M}(C) = (2n+1)(n+1)^2 \int_0^1 \frac{dt}{(n + (t^\theta + (1-t)^\theta)^{1/\theta})^2} - (2n+1).$$

Example 2. If we take

$$f(t) = 2 - 2\log 2 - \int_0^1 \frac{1}{w} \log(1+w) dw + \log(1+t).$$

Then we see that

$$\begin{aligned} \mathcal{M}(C) &= 2 - 2\log 2 - \int_0^1 \frac{1}{s} \log(1+s) ds \\ &\quad + \int_0^1 \log(1+s) ds - \int_0^1 \frac{s}{1+s} ds - \iint_{\mathbb{I}^2} \frac{1}{(1+s)^2} s^{A(t)} \log s \, dsdt \\ &= \log 2 - \int_0^1 \frac{1}{s} \log(1+s) ds - \iint_{\mathbb{I}^2} \frac{1}{(1+s)^2} s^{A(t)} \log s \, dsdt. \end{aligned}$$

5. Discussions

We have developed a generalization of a measure of concordance for the class of extreme value copulas which are given through the limiting procedure and are related to the modeling of extreme events. Utilizing the representation formula, we are able to obtain a generalized formula for the measure of association. Examples show that with this formula the computation becomes rather straightforward. We hope that our methodology may serve as a handy tool of comparing the effect of dependence model.

There remain several points to be discussed further; one of these is the pursuit of more generalization. This will be interesting both from theoretical and practical points of view.

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6 Constructing extreme value copula densities from given copulas. 7 Extensions to nested Archimedean copulas. 8 Copulas of extreme-value distributions. 9 Conclusion. References. A Implementation of various Pickand's dependence functions and their first and second order derivatives. B Code for constructing extreme value copula densities from Archimedean copulas. UniversitÃ¤t ZÃ¼rich. ETH ZÃ¼rich. On Densities of Extreme Value Copulas. M.Sc. Thesis.