

# A Course in Model Theory I:

## Introduction<sup>1</sup>

Rami Grossberg

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213

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<sup>1</sup>This **preliminary draft** is dated from August 14, 2019. The book will be published by Cambridge University Press. The book is approximately 98.52% complete. I expect that the final version will have about 800 pages, many sections of the current version will be revised and few will be added. I hope to have a stable version of this volume soon.

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2000. Topics. Model theory. Publisher. New York : Springer. The authors' development of IRT builds on the foundations of classical test theory, nonlinear factor analysis, and generalized linear models. Using these foundational concepts, the authors then explain IRT models, estimation via maximum likelihood, item characteristic curves, and information functions. It is a useful text for IRT courses and a good resource for researchers who use IRT. About the authors. Tenko Raykov is Professor of Measurement and Quantitative Methods at Michigan State University. He specializes in latent-variable and structural equation modeling, multivariate statistics, item response theory and modeling, missing data analysis, multilevel modeling, scale construction and development, longitudinal data modeling, survival analysis, and applied statistics.